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論篩法及其有關的若干應用 (I)*

——表大整數爲殆素數**之和——

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1. 結果的陳述

本文的宗旨在於證明作者在[1] 內所提及的非條件結果及 [2] 內所提及的全部結果. 爲簡單起見,將下面的命題記爲 (a, b):

每一充分大的偶數可表爲兩個大於 1 的整數 c_1 與 c_2 之和, c_1 與 c_2 的素因子個數(包含相同的與相異的)分別不超過 a 與 b.

並不需要很複雜的數值計算,就能得到 $(3, 3)$ 及 (a, b), $(a + b \leqslant 5)$. 用比較複雜的數值計算,我們得到了 $(2, 3)$. 另一點值得注意的是本文所用的方法完全是初等的,而 A. И. Виноградов[3,4] 在證明(3,3)的過程中却引用了精深的 Riemann-ζ 函數論的結果.

命 $P(M, \xi)$ 表示給定有限整數集合 M 中不被 $\leqslant \xi$ 的素數所整除的元素的個數. 本文還附帶舉例以討論 A. Selberg[5] 所提出的關於估計 $P(M, \xi)$ 下界的方法的某些限度問題.

現在把本文的結果詳述於下:

定理 1. $(2, 3)$.

關於表示大奇數的問題及孿生素數問題,我們亦得到:

定理 2. 對於任何偶數 k, 皆存在無限多個整數 n 使: (i) n 與 $n + k$ 的素因子個數均不超過 3, (ii) $n(n + k)$ 爲不超過 5 個素數的乘積.

定理 3. 每一充分大的奇數可表爲 $2N + 1 = 2p_1 \cdots p_c + q_1 \cdots q_d$, 此處 c, d 滿足 $1 \leqslant c \leqslant 3, 1 \leqslant d \leqslant 3$ 及 $c + d \leqslant 5$, 而 p_i, q_j 均爲素數.

本文中之 $p; p', p'', \cdots; p_1, p_2, \cdots$ 均表示素數,不再一一聲明.

最後,作者衷心感謝華羅庚教授的鼓勵與指導.

2. 計算

引 1. 命 $\Omega(n)$ 表示 n 的不同的素因子的個數. 若 $x \geqslant 1, z \geqslant 1$, 則

$$\sum_{\substack{n \leqslant z \\ (n,x)=1}} \frac{|\mu(n)| 2^{\Omega(n)}}{n} = \frac{1}{2} \prod_{p|x} \frac{p}{p+2} \prod_{p} \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p}\right) \cdot \log^2 z + O(\log 2z \cdot \cdot \log\log 3zx) + O((\log\log 3x)^2),$$

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** 殆素數者即素因子個數不超過某一確定限的整數.

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此處 $\mu(n)$ 表示熟知的 Möbius 函數.

證明見 Shapiro 與 Warga[6].

引 2. 命 $2|x$. 若 $z\geqslant 1$, 則

$$\sum_{\substack{n\leqslant z\\(n,x)=1}}\frac{|\mu(n)|2^{\Omega(n)}}{n}\prod_{p|n}\left(1+\frac{2}{p-2}\right)=\frac{1}{8}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|x\\p>2}}\frac{p-2}{p}\log^2 z+ + O(\log xz\log\log 3xz).$$

證. 命 $\Psi(r)=\prod_{p|r}(p-2)$. 則由引 1 得

$$\sum_{\substack{n\leqslant z\\(n,x)=1}}\frac{|\mu(n)|2^{\Omega(n)}}{n}\prod_{p|n}\left(1+\frac{2}{p-2}\right)=\sum_{\substack{n\leqslant z\\(n,x)=1}}\frac{|\mu(n)|2^{\Omega(n)}}{n}\sum_{r|n}\frac{2^{\Omega(r)}}{\Psi(r)}=$$

$$=\sum_{\substack{r\leqslant z\\(r,x)=1}}\frac{|\mu(r)|2^{2\Omega(r)}}{r\Psi(r)}\sum_{\substack{s\leqslant z/r\\(s,rx)=1}}\frac{|\mu(s)|2^{\Omega(s)}}{s}=$$

$$=\sum_{\substack{r\leqslant z\\(r,x)=1}}\frac{|\mu(r)|2^{2\Omega(r)}}{r\Psi(r)}\left\{\frac{1}{2}\prod_{p}\frac{(p-1)^2(p+2)}{p^3}\prod_{p|rx}\frac{p}{p+2}\log^2\frac{z}{r}+ + O(\log zx\log\log 3zx)\right\}=$$

$$=\frac{1}{2}\prod_{p}\frac{(p-1)^2(p+2)}{p^3}\prod_{p|x}\frac{p}{p+2}\log^2 z\cdot\sum_{\substack{r\leqslant z\\(r,x)=1}}\frac{|\mu(r)|4^{\Omega(r)}}{\prod_{p|r}(p^2-4)}+ + O(\log zx\log\log 3zx)=$$

$$=\frac{1}{8}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|x\\p>2}}\frac{p-2}{p}\log^2 z+O(\log xz\log\log 3xz).$$

引理證完.

引 3. 對於任何 $\eta>0$, 皆存在 $x_0=x_0(\eta)$, 當 $x>x_0$ 時,

$$2^{\Omega(x)}\leqslant d(x)\leqslant 2^{(1+\eta)\frac{\log x}{\log\log x}},$$

此處 $d(x)$ 表示 x 的因子個數.

證明略去.

引 4. 當 $x\geqslant 1$ 時, 命 $\Delta(x)=e^{\frac{\log 3x}{\log\log 9x}}$. 則存在 $c_1>0$, 使

$$\prod_{p|x}\left(1-\frac{1}{p}\right)^{-1}-\left\{1+\sum_{\substack{p|x\\p\leqslant\Delta(x)}}\frac{1}{p}\left(1-\frac{1}{p}\right)^{-1}+\sum_{\substack{pp'|x\\pp'\leqslant\Delta(x)\\p\neq p'}}\frac{1}{pp'}\left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p'}\right)^{-1}+\cdots\right\}=$$

$$=O\left(e^{-c_1\frac{\log 3x}{\log\log 9x}}\right).$$

證. 由引 3 可知

$$\prod_{p|x}\left(1-\frac{1}{p}\right)^{-1}-\left\{1+\sum_{\substack{p|x\\p\leqslant\Delta(x)}}\frac{1}{p}\left(1-\frac{1}{p}\right)^{-1}+\sum_{\substack{pp'|x\\pp'\leqslant\Delta(x)\\p\neq p'}}\frac{1}{pp'}\left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p'}\right)^{-1}+\cdots\right\}=$$

$$=\sum_{\substack{p|x\\p>\Delta(x)}}\frac{1}{p}\left(1-\frac{1}{p}\right)^{-1}+\sum_{\substack{pp'|x\\pp'>\Delta(x)\\p\neq p'}}\frac{1}{pp'}\left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p'}\right)^{-1}+\cdots\leqslant$$

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$$\leqslant \frac{1}{\Delta(x)} \prod_{p|x}\left(1-\frac{1}{p}\right)^{-1}\Big\{1+\sum_{p|x}1+\sum_{\substack{pp'|x\\ p\neq p'}}1+\cdots\Big\}=$$

$$=\frac{2^{\Omega(x)}}{\Delta(x)\prod_{p|x}\left(1-\frac{1}{p}\right)}=$$

$$=O\left(e^{-c_1\frac{\log 3x}{\log\log 9x}}\right).$$

注意，在此用到估計：以 p_i 表第 i 個素數，

$$\prod_{p|x}\left(1-\frac{1}{p}\right)^{-1}=O\Big(\prod_{i\leqslant\Omega(x)}\Big(1-\frac{1}{p_i}\Big)^{-1}\Big)=O\Big(\prod_{p\leqslant c\log 2x}\Big(1-\frac{1}{p}\Big)^{-1}\Big)=O(\log\log 3x).$$

引理證完.

引 5. 命 q 爲給定偶數，給予整數 y. 命 $g(1)=1$, $g(p)=\frac{1}{p}\,(p|y)$, $g(p)=\frac{2}{p}\,(p\nmid y)$，若 n 無平方因子，則命 $g(n)=\prod_{p|n}g(p)$, $f(n)=\frac{1}{g(n)}\prod_{p|n}(1-g(p))$. 若 $z\geqslant 1$，則

$$\sum_{\substack{n\leqslant z\\ (n,q)=1}}\frac{|\mu(n)|}{f(n)}=\frac{1}{4}\prod_{p|q}\frac{p-1}{p}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|qy\\ p>2}}\frac{p-2}{p-1}\log^2 z+O\Big(\prod_{\substack{p|qy\\ p>2}}\frac{p-2}{p-1}\cdot\frac{\log^2 2yz}{\log\log 3yz}\Big).$$

證. 分三種情况證之.

(i) 若 $z\geqslant\Delta(y)\geqslant\log 2z$，則由引 2，引 4 得

$$\sum_{\substack{n\leqslant z\\ (n,q)=1}}\frac{|\mu(n)|}{f(n)}=\sum_{\substack{n\leqslant z\\ (n,qy)=1}}\frac{|\mu(n)|2^{\Omega(n)}}{n}\prod_{p|n}\left(1-\frac{2}{p}\right)^{-1}+$$

$$+\sum_{\substack{p'|y\\ p'\nmid q}}\frac{1}{p'}\left(1-\frac{1}{p'}\right)^{-1}\sum_{\substack{n\leqslant z/p'\\ (n,qy)=1}}\frac{|\mu(n)|2^{\Omega(n)}}{n}\prod_{p|n}\left(1-\frac{2}{p}\right)^{-1}+$$

$$+\sum_{\substack{p'p''|y\\ (p'p'',q)=1\\ p'\neq p''}}\frac{1}{p'p''}\left(1-\frac{1}{p'}\right)^{-1}\left(1-\frac{1}{p''}\right)^{-1}\sum_{\substack{n\leqslant z/p'p''\\ (n,qy)=1}}\frac{|\mu(n)|2^{\Omega(n)}}{n}\prod_{p|n}\left(1-\frac{2}{p}\right)^{-1}+\cdots\geqslant$$

$$\geqslant\Big\{1+\sum_{\substack{p'|y\\ p'\leqslant\Delta(y)\\ p'\nmid q}}\frac{1}{p'}\left(1-\frac{1}{p'}\right)^{-1}+$$

$$+\sum_{\substack{p'p''|y\\ (p'p'',q)=1\\ p'\neq p''\\ p'p''\leqslant\Delta(y)}}\frac{1}{pp'}\left(1-\frac{1}{p'}\right)^{-1}\left(1-\frac{1}{p''}\right)^{-1}+\cdots\Big\}\sum_{\substack{n\leqslant z/\Delta(y)\\ (n,qy)=1}}\frac{|\mu(n)|2^{\Omega(n)}}{n}\prod_{p|n}\left(1+\frac{2}{p-2}\right)=$$

$$=\Big(\prod_{\substack{p|y\\ p\nmid q}}\left(1-\frac{1}{p}\right)^{-1}+O\Big(e^{-c_1\frac{\log 3y}{\log\log 9y}}\Big)\Big)\Big(\frac{1}{8}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|qy\\ p>2}}\frac{p-2}{p}\log^2\frac{z}{\Delta(y)}+$$

$$+O(\log 2zy\log\log 3zy)\Big)=$$

$$=\frac{1}{4}\prod_{p|q}\frac{p-1}{p}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|qy\\ p>2}}\frac{p-2}{p-1}\log^2 z+O\Big(\prod_{\substack{p|qy\\ p>2}}\frac{p-2}{p-1}\cdot\frac{\log 2z\log 2y}{\log\log 3y}\Big)+$$

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$$+O\left(e^{-c_1\frac{\log 3y}{\log\log 9y}}\log^2 2yz\right)+O\left(\frac{\log^2 2y}{(\log\log 3y)^2}\right)+O(\log 2zy(\log\log 3zy)^2)=$$

$$=\frac{1}{4}\prod_{p|q}\frac{p-1}{p}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\log^2 z+O\left(\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\cdot\frac{\log^2 2zy}{\log\log 3zy}\right).$$

注意此處用到 $\log 2y\gg\log\log 3z$, $\log\log 3z\gg\log\log 3y$ (由 $z\geqslant\Delta(y)\geqslant\log 2z$ 推出) 及當 $y>y_0$ 時, 函數 $\frac{\log 2y}{\log\log 3y}$ 的遞增性.

另一方面, 可知

$$\sum_{\substack{n\leqslant z\\(n,q)=1}}\frac{|\mu(n)|}{f(n)}\leqslant\left\{1+\sum_{\substack{p'|y\\p'\nmid q}}\frac{1}{p'}\left(1-\frac{1}{p'}\right)^{-1}+\right.$$

$$\left.+\sum_{\substack{p'p''|y\\(p'p'',q)=1\\p'\neq p''}}\frac{1}{p'p''}\left(1-\frac{1}{p'}\right)^{-1}\left(1-\frac{1}{p''}\right)^{-1}+\cdots\right\}\sum_{\substack{n\leqslant z\\(n,qy)=1}}\frac{|\mu(n)|2^{\vartheta(n)}}{n}\prod_{p|n}\left(1-\frac{2}{p}\right)^{-1}=$$

$$=\prod_{\substack{p|y\\p\nmid q}}\left(1-\frac{1}{p}\right)^{-1}\left(\frac{1}{8}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p}\log^2 z+O(\log 2zy\log\log 3zy)\right)=$$

$$=\frac{1}{4}\prod_{p|q}\frac{p-1}{p}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\log^2 z+O\left(\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\cdot\frac{\log^2 2zy}{\log\log 3zy}\right).$$

綜合上述, 乃得引理.

(ii) 若 $\Delta(y)<\log 2z$, 則 $y<e^{c(\log\log 3z)^2}$ $(c>0)$. 與 (i) 相仿可知:

$$\sum_{\substack{n\leqslant z\\(n,q)=1}}\frac{|\mu(n)|}{f(n)}\geqslant\left\{1+\sum_{\substack{p'|y\\p'\nmid q}}\frac{1}{p'}\left(1-\frac{1}{p'}\right)^{-1}+\right.$$

$$\left.+\sum_{\substack{p'p''|y\\(p'p'',q)=1\\p'\neq p''}}\frac{1}{p'p''}\left(1-\frac{1}{p'}\right)^{-1}\left(1-\frac{1}{p''}\right)^{-1}+\cdots\right\}\sum_{\substack{n\leqslant z/e^{c(\log\log 3z)^2}\\(n,qy)=1}}\frac{|\mu(n)|2^{\vartheta(n)}}{n}\prod_{p|n}\left(1-\frac{2}{p}\right)^{-1}=$$

$$=\frac{1}{4}\prod_{p|q}\frac{p-1}{p}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\log^2 z+O\left(\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\frac{\log^2 2zy}{\log\log 3zy}\right).$$

及

$$\sum_{\substack{n\leqslant z\\(n,q)=1}}\frac{|\mu(n)|}{f(n)}\leqslant\left\{1+\sum_{\substack{p'|y\\p'\nmid q}}\frac{1}{p'}\left(1-\frac{1}{p'}\right)^{-1}+\right.$$

$$\left.+\sum_{\substack{p'p''|y\\(p'p'',q)=1\\p'\neq p''}}\frac{1}{p'p''}\left(1-\frac{1}{p'}\right)^{-1}\left(1-\frac{1}{p''}\right)^{-1}+\cdots\right\}\sum_{\substack{n\leqslant z\\(n,qy)=1}}\frac{|\mu(n)|2^{\vartheta(n)}}{n}\prod_{p|n}\left(1-\frac{2}{p}\right)^{-1}=$$

$$=\frac{1}{4}\prod_{p|q}\frac{p-1}{p}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\log^2 z+O\left(\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\frac{\log^2 2zy}{\log\log 3zy}\right).$$

亦得引理.

(iii) 若 $\Delta(y)>z$, 則由引 2 得

$$\sum_{\substack{n\leqslant z\\(n,q)=1}}\frac{|\mu(n)|}{f(n)}=O\left(\sum_{\substack{n\leqslant z\\(n,q)=1}}\frac{|\mu(n)|2^{\vartheta(n)}}{n}\prod_{p|n}\left(1-\frac{2}{p}\right)^{-1}\right)=O(\log^2 2z)=$$

$$=O\left(\frac{\log^2 2y}{\log\log 3y}\right)=O\left(\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\cdot\frac{\log^2 2zy}{\log\log 3zy}\right),$$

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及

$$\frac{1}{4}\prod_{p|q}\frac{p-1}{p}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\log^2 z = O(\log^2 2z) =$$

$$= O\Big(\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\cdot\frac{\log^2 2zy}{\log\log 3zy}\Big).$$

故得

$$\sum_{\substack{n\leqslant z\\(n,q)=1}}\frac{|\mu(n)|}{f(n)} = \frac{1}{4}\prod_{p|q}\frac{p-1}{p}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\log^2 z +$$

$$+ O\Big(\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\cdot\frac{\log^2 2zy}{\log\log 3zy}\Big).$$

綜合 (i), (ii), (iii), 乃得引理.

引 6.　命 $\beta>\alpha>1$ 是固定兩數,則當 $x\geqslant 2$ 時

$$\sum_{x^{\frac{1}{\beta}}<p\leqslant x^{\frac{1}{\alpha}}}\frac{1}{p\log^2\frac{x}{p}} = \frac{1}{\log^2 x}\Big\{\log\frac{\beta-1}{\alpha-1}+\frac{1}{\alpha-1}-\frac{1}{\beta-1}\Big\}+O\Big(\frac{1}{\log^3 x}\Big).$$

證明參看 Бухштаб[7].

3.　定理 A

給予兩數 $2\leqslant y\leqslant x$; 給出一組整數

(ω)　　$a, q; a_i, b_i\ (1\leqslant i\leqslant r),$

滿足

(1)　　$2|q,\ q=O(1)$; 若 $p_i|y$, 則 $a_i\equiv b_i \pmod{p_i}$, 否則,

$$a_i\not\equiv b_i \pmod{p_i}\ (i=1, 2, \cdots, r),$$

此處 $2<p_1<\cdots<p_r\leqslant\xi$ 爲不超過 ξ 而又不能整除 q 的全部素數,而 $\xi>q$.

命 $P_\omega(x, q, \xi)$ 爲適合下面條件的整數 n 的個數:

(2)　$1\leqslant n\leqslant x,\ n\equiv a \pmod q,\ n\not\equiv a_i \pmod{p_i},\ n\not\equiv b_i \pmod{p_i},\ (1\leqslant i\leqslant r),$

由孫子定理可知同餘式組

$$y\equiv a_i \pmod{p_i}\ (1\leqslant i\leqslant r),\ y\equiv b_i \pmod{p_i},\ (1\leqslant i\leqslant r)$$

均在區間 $1\leqslant y\leqslant p_1\cdots p_r$ 內有唯一的解. 分別記之爲 a^* 及 b^*. 可知適合 (2) 式與適合下式的整數個數相同:

(3)　$1\leqslant n\leqslant x,\ n\equiv a \pmod q,\ (n-a^*)(n-b^*)\not\equiv 0 \pmod{p_i},\ (1\leqslant i\leqslant r).$

定理 A.　命 $c>0$, $P=\prod_{i=1}^{r}p_i$. 則對於任意給予的整數列 (ω), 下式一致成立:

$$P_\omega(x, q, \xi)\leqslant\frac{x}{q\sum_{\substack{m\leqslant\xi^c\\m|P}}\frac{|\mu(m)|}{f(m)}}+O(\xi^{2c}\log^6\xi),$$

此處 $g(1)=1$, $g(p)=\frac{1}{p}\ (p|y)$, $g(p)=\frac{2}{p}\ (p\nmid y)$, 當 n 無平方因子時, $g(n)=$

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$$= \prod_{p|n} g(p),\ f(n) = \sum_{d|n} \frac{\mu(d)}{g\left(\frac{n}{d}\right)} = \frac{1}{g(n)} \prod_{p|n} (1 - g(p)).$$

證. 當 $k|P$ 時,由於 $(k, q) = 1$, 故由孫子定理可知同餘式組

$$\begin{cases} (n - a^*)(n - b^*) \equiv 0 \pmod k \\ n \equiv a \pmod q \end{cases}$$

在區間 $1 \leqslant n \leqslant kq$ 中的解數爲 $2^{\Omega(k)-\Omega((k,q))}$. 因此

$$\sum_{\substack{k|(n-a^*)(n-b^*) \\ n\equiv a \pmod q \\ 1\leqslant n\leqslant x}} 1 = 2^{\Omega(k)-\Omega((k,q))} \left[\frac{x}{kq}\right] + O(2^{\Omega(k)}) = g(k)\frac{x}{q} + O(2^{\Omega(k)}).$$

當 $k|P$ 時,命

$$\lambda_k = \frac{\mu(k)}{g(k)f(k)} \sum_{\substack{1\leqslant n\leqslant \xi^c/k \\ (n,k)=1 \\ n|P}} \frac{|\mu(n)|}{f(n)} \Bigg/ \sum_{\substack{1\leqslant l\leqslant \xi^c \\ l|P}} \frac{|\mu(l)|}{f(l)}.$$

由於 $\lambda_1 = 1$, $\lambda_d = 0\ (d > \xi^c)$ 及滿足條件(2)與條件(3)的整數 n 的個數相同,故

$$P_\omega(x, q, \xi) = \sum_{\substack{1\leqslant n\leqslant x \\ ((n-a^*)(n-b^*),\ P)=1 \\ n\equiv a \pmod q}} 1 \leqslant \sum_{\substack{n\leqslant x \\ n\equiv a \pmod q}} \Bigg(\sum_{d|(n-a^*)(n-b^*)} \lambda_d\Bigg)^2 =$$

$$= \sum_{\substack{d_1\leqslant \xi^c \\ d_1|P}} \sum_{\substack{d_2\leqslant \xi^c \\ d_2|P}} \lambda_{d_1}\lambda_{d_2} \sum_{\substack{\{d_1,d_2\}|(n-a^*)(n-b^*) \\ 1\leqslant n\leqslant x \\ n\equiv a \pmod q}} 1 =$$

$$= \frac{x}{q} \sum_{\substack{d_1\leqslant \xi^c \\ d_1|P}} \sum_{\substack{d_2\leqslant \xi^c \\ d_2|P}} \lambda_{d_1}\lambda_{d_2} g(\{d_1, d_2\}) +$$

$$+ O\Bigg(\sum_{\substack{d_1\leqslant \xi^c \\ d_1|P}} \sum_{\substack{d_2\leqslant \xi^c \\ d_2|P}} |\lambda_{d_1}\lambda_{d_2}| 2^{\Omega(d_1)+\Omega(d_2)}\Bigg) =$$

$$= \frac{x}{q} Q + R,$$

此處 $\{d_1, d_2\}$ 表示 d_1 與 d_2 的最小公倍數. 與王元[8]相同,可知

$$Q = \frac{1}{\sum\limits_{\substack{n\leqslant \xi^c \\ n|P}} \frac{|\mu(n)|}{f(n)}}.$$

又

$$R = O\Bigg(\Bigg(\sum_{\substack{k\leqslant \xi^c \\ k|P}} |\lambda_k| 2^{\Omega(k)}\Bigg)^2\Bigg) = O\Bigg(\Bigg(\sum_{\substack{k\leqslant \xi^c \\ k|P}} \frac{|\mu(k)|}{g(k)f(k)} 2^{\Omega(k)}\Bigg)^2\Bigg) =$$

$$= O\Bigg(\Bigg(\sum_{\substack{1\leqslant k\leqslant \xi^c \\ k|P}} \frac{|\mu(k)| 2^{\Omega(k)}}{\prod\limits_{2<p\leqslant \xi^c}\left(1 - \frac{2}{p}\right)}\Bigg)^2\Bigg) = O\Bigg(\log^4 \xi \Bigg(\sum_{k\leqslant \xi^c} d(k)\Bigg)^2\Bigg) =$$

$$= O(\xi^{2c} \log^6 \xi).$$

明所欲證.

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4. 定理 A 的應用

如前段之記號，不一一解釋.

當 $l < c \leqslant l+1$ 時(l 爲正整數)，用逐步淘汰原則得

(4) $$\sum_{\substack{m\leqslant\xi^c\\ m|P}}\frac{|\mu(m)|}{f(m)}=\sum_{\substack{m\leqslant\xi^c\\ (m,q)=1}}\frac{|\mu(m)|}{f(m)}-\sum_{\xi<p<\xi^c}\frac{1}{f(p)}\sum_{\substack{m\leqslant\xi^c/p\\ (m,qp)=1}}\frac{|\mu(m)|}{f(m)}+$$
$$+\sum_{\substack{\xi<p<p'\\ pp'\leqslant\xi^c}}\frac{1}{f(p)f(p')}\sum_{\substack{m\leqslant\xi^c/pp'\\ (m,qpp')=1}}\frac{|\mu(m)|}{f(m)}+$$
$$+\cdots+(-1)^l\sum_{\substack{\xi<p'<\cdots<p^{(l)}\\ p'p''\cdots p^{(l)}\leqslant\xi^c}}\frac{1}{f(p')\cdots f(p^{(l)})}\sum_{\substack{m\leqslant\xi^c/p'\cdots p^{(l)}\\ (m,qp'\cdots p^{(l)})=1}}\frac{|\mu(m)|}{f(m)}.$$

1°. 當 $1\leqslant c\leqslant 2$ 時，取 x 充分大，又取 $\xi=\frac{x^{\frac{1}{2c}}}{\log^5 x}(>q)$. 記 $2c=d$. 由於

$$\sum_{\xi<p\leqslant\xi^c}\frac{1}{f(p)}-\sum_{\xi<p\leqslant\xi^c}\frac{2}{p}=O\left(\sum_{\substack{\xi<p\leqslant\xi^c\\ p\nmid y}}\frac{1}{p^2}\right)+O\left(\sum_{\substack{p|y\\ p>\xi}}\frac{1}{p}\right)=O\left(\frac{1}{\xi}\right)=O\left(\frac{\log^5 x}{x^{\frac{1}{d}}}\right),$$

故由(4)式及引 5 可知

$$\sum_{\substack{m\leqslant\xi^c\\ m|P}}\frac{|\mu(m)|}{f(m)}=\sum_{\substack{m\leqslant\xi^c\\ (m,q)=1}}\frac{|\mu(m)|}{f(m)}-\sum_{\xi<p\leqslant\xi^c}\frac{2}{p}\sum_{\substack{m\leqslant\xi^c/p\\ (m,q)=1}}\frac{|\mu(m)|}{f(m)}+O\left(\frac{\log^7 x}{x^{\frac{1}{d}}}\right)=$$
$$=\left\{(d-1)^2-2\left(\frac{d}{2}\right)^2\log\frac{d}{2}\right\}x\frac{1}{4}\prod_{p|q}\frac{p-1}{p}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{\substack{p|qy\\ p>2}}\frac{p-2}{p-1}\log^2\xi+$$
$$+O\left(\prod_{\substack{p|qy\\ p>2}}\frac{p-2}{p-1}\frac{\log^2 x}{\log\log x}\right).$$

故由定理A可知

(5) $$P_\omega(x,q,x^{\frac{1}{d}})\leqslant P_\omega\left(x,q,\frac{x^{\frac{1}{d}}}{\log^5 x}\right)\leqslant\Lambda(d)c_{qy}\frac{x}{\log^2 x}+O\left(\prod_{\substack{p|qy\\ p>2}}\frac{p-2}{p-1}\frac{\log^2 x}{\log\log x}\right),$$

此處

(6) $$\Lambda(d)=2e^{2\gamma}\left[\frac{d^2}{(d-1)^2-2\left(\frac{d}{2}\right)^2\log\frac{d}{2}}\right]\quad(2\leqslant d\leqslant 4),$$

其中

(7) $$c_{qy}=\frac{2e^{-2\gamma}}{q}\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|q}\frac{p}{p-1}\prod_{\substack{p|qy\\ p>2}}\frac{p-1}{p-2},$$ γ 爲 Euler 常數.

2°. 當 $2\leqslant c\leqslant 3$，取 $\xi=\frac{x^{\frac{1}{2c}}}{\log^5 x}(>q)$. 記 $2c=d$. 由於

$$\sum_{\xi<p<\xi^c}\frac{1}{f(p)}\sum_{\substack{n\leqslant\xi^c/p\\ (n,q)=1}}\frac{|\mu(n)|}{f(n)}-\sum_{\xi<p<\xi^c}\frac{1}{f(p)}\sum_{\substack{n\leqslant\xi^c/p\\ (n,qp)=1}}\frac{|\mu(n)|}{f(n)}=O\left(\frac{\log^7 x}{x^{\frac{1}{d}}}\right)$$

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及

$$\sum_{\substack{\xi<p<p'\\pp'\leqslant\xi^c}}\frac{1}{f(p)f(p')}-\sum_{\substack{\xi<p<p'\\pp'\leqslant\xi^c}}\frac{4}{pp'}=O\left(\frac{\log^5 x}{x^{\frac{1}{d}}}\right).$$

故由(4)式及引5可知

$$\sum_{\substack{n\leqslant\xi^c\\n|P}}\frac{|\mu(n)|}{f(n)}=\sum_{\substack{n\leqslant\xi^c\\(n,q)=1}}\frac{|\mu(n)|}{f(n)}-\sum_{\xi<p\leqslant\xi^c}\frac{2}{p}\sum_{\substack{n\leqslant\xi^c/p\\(n,q)=1}}\frac{|\mu(n)|}{f(n)}+\sum_{\substack{\xi<p<p'\\pp'\leqslant\xi^c}}\frac{4}{pp'}\sum_{\substack{n\leqslant\xi^c/pp'\\(n,q)=1}}\frac{|\mu(n)|}{f(n)}+$$

$$+O\left(\frac{\log^7 x}{x^{1/d}}\right)=$$

$$=\left\{(d-1)^2-2\left(\frac{d}{2}\right)^2\log\frac{d}{2}+\frac{4}{\log^2\xi}\sum_{\substack{\xi<p<p'\\pp'\leqslant\xi^c}}\frac{1}{pp'}\log^2\frac{\xi^c}{pp'}\right\}\cdot$$

$$\cdot\frac{1}{4}\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{p|q}\frac{p-1}{p}\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\log^2\xi+$$

$$+O\left(\prod_{\substack{p|qy\\p>2}}\frac{p-2}{p-1}\cdot\frac{\log^2 x}{\log\log x}\right).$$

故由定理A得知對於任何 $\varepsilon>0$，皆存在 $x_0=x_0(\varepsilon)$，當 $x>x_0$ 時，仍得(5)式，但

$$\Lambda(d)=2e^{2\gamma}\left(\frac{d^2}{(d-1)^2-2\left(\frac{d}{2}\right)^2\log\frac{d}{2}+\delta\left(\frac{d}{2}\right)-\varepsilon}\right)\quad(4\leqslant d\leqslant 6),\tag{8}$$

此處

$$\delta(\alpha)=\lim_{\xi\to\infty}\frac{4}{\log^2\xi}\sum_{\substack{\xi<p<p'\\pp'\leqslant\xi^\alpha}}\frac{1}{pp'}\log^2\frac{\xi^\alpha}{pp'}\quad(\alpha\geqslant 2).\tag{9}$$

同法可知當 $6\leqslant d\leqslant 8$ 時，(5)式當 $x>x_1(\varepsilon)$ 時成立，但

$$\Lambda(d)=2e^{2\gamma}\left(\frac{d^2}{(d-1)^2-2\left(\frac{d}{2}\right)^2\log\frac{d}{2}+\delta\left(\frac{d}{2}\right)-K\left(\frac{d}{2}\right)-\varepsilon}\right)\quad(6\leqslant d\leqslant 8),\tag{10}$$

此處

$$K(\alpha)=\overline{\lim_{\xi\to\infty}}\frac{8}{\log^2\xi}\sum_{\substack{pp'p''\leqslant\xi\\\xi<p<p'<p''}}\frac{1}{pp'p''}\log^2\frac{\xi^\alpha}{pp'p''}\quad(\alpha\geqslant 3).\tag{11}$$

以下可以同法依次類推．又當 $0<d\leqslant 2$ 時，置 $\Lambda(d)=\Lambda(2)$．則(5)顯然成立．

下面我們提供一個簡單估計 $\delta(\alpha)$ 與 $K(\alpha)$ 的方法．例如：求 $\delta(3)$．吾人先求出 $\underline{\lim}_{\xi\to\infty}\sum_{\substack{\xi<p<p'\\\xi^{2.1}<pp'\leqslant\xi^{2.2}}}\frac{1}{pp'}$ 等．

$$\underline{\lim}_{\xi\to\infty}\sum_{\substack{\xi<p<p'\\\xi^{2.1}<pp'\leqslant\xi^{2.2}}}\frac{1}{pp'}\geqslant\underline{\lim}_{\xi\to\infty}\left(\sum_{\xi<p<\xi^{1.01}}\frac{1}{p}\sum_{\xi^{1.1}<p'<\xi^{1.19}}\frac{1}{p'}+\right.$$

$$+\sum_{\xi^{1.01}<p<\xi^{1.02}}\frac{1}{p}\sum_{\xi^{1.09}<p'<\xi^{1.18}}\frac{1}{p'}+\cdots+$$

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$$+\cdots+\sum_{\xi^{1.08}<p<\xi^{1.09}}\frac{1}{p}\sum_{\xi^{1.09}<p'<\xi^{1.11}}\frac{1}{p'}+\sum_{\xi^{1.09}<p<\xi^{1.095}}\frac{1}{p}\sum_{\xi^{1.095}<p'<\xi^{1.105}}\frac{1}{p'}\Big)=$$

$$=\log 1.01\log\frac{1.19}{1.1}+\log\frac{1.02}{1.01}\log\frac{1.18}{1.09}+\cdots+$$

$$+\log\frac{1.09}{1.08}\log\frac{1.11}{1.09}+\log\frac{1.095}{1.09}\log\frac{1.105}{1.095}>0.00561.$$

$$\delta(3)\geqslant 4\lim_{\xi\to\infty}\Big(0.81\sum_{\substack{\xi<p<p'\\ pp'\leqslant\xi^{2.1}}}\frac{1}{pp'}+0.64\sum_{\substack{\xi<p<p'\\ \xi^{2.1}<pp'\leqslant\xi^{2.2}}}\frac{1}{pp'}+\cdots+0.01\sum_{\substack{\xi<p<p'\\ \xi^{2.8}<pp'\leqslant\xi^{2.9}}}\frac{1}{pp'}\Big)>$$

$$>0.087202.$$

5. 定理 B

定理 B$_1$. 命 $1<\alpha<\beta\leqslant 15$ 爲固定兩數. 若有非負遞增且僅有有限多個不連續點之函數 $\Lambda(z)$ $(0<z\leqslant 14)$ 使

$$P_\omega(x,q,x^{\frac{1}{z}})<\Lambda(z)\frac{c_{qy}x}{\log^2 x}+O\Big(\prod_{\substack{p\mid qy\\ p>2}}\frac{p-1}{p-2}\cdot\frac{x}{\log^2 x\log\log x}\Big),$$

此處與"O"有關的常數爲絕對常數,則

$$\sum_{\substack{x^{\frac{1}{\beta}}<p\leqslant x^{\frac{1}{\alpha}}\\ p\nmid y}}P_{\omega_p}\Big(\frac{x}{p},q,x^{\frac{1}{\beta}}\Big)\leqslant\Big(\int_{\alpha-1}^{\beta-1}\Lambda\Big(\frac{\beta z}{z+1}\Big)\frac{z+1}{z^2}dz\Big)\frac{c_{qy}x}{\log^2 x}+O\Big(\frac{c_{qy}x}{\log^2 x\log\log x}\Big),$$

此處與"O"有關之常數與諸 (ω_p) 無關.

證. 命 $n=[\sqrt{\log x}]$, $u_l=\alpha+\dfrac{\beta-\alpha}{n}l-1$ $(0\leqslant l\leqslant n)$. 則由引 6 得

$$T_l=\sum_{\substack{x^{\frac{1}{u_{l+1}+1}}<p\leqslant x^{\frac{1}{u_l+1}}\\ p\nmid y}}P_{\omega_p}\Big(\frac{x}{p},q,x^{\frac{1}{\beta}}\Big)=\sum_{\substack{x^{\frac{1}{u_{l+1}+1}}<p\leqslant x^{\frac{1}{u_l+1}}\\ p\nmid y}}P_{\omega_p}\Big(\frac{x}{p},q,\Big(\frac{x}{p}\Big)^{\frac{\log x}{\beta\log\frac{x}{p}}}\Big)\leqslant$$

$$\leqslant\sum_{x^{\frac{1}{u_{l+1}+1}}<p\leqslant x^{\frac{1}{u_l+1}}}\left\{\Lambda\Big(\frac{\beta\log\frac{x}{p}}{\log x}\Big)\frac{c_{qy}x}{p\log^2\frac{x}{p}}+O\Big(\frac{c_{qy}x}{p\log^2\frac{x}{p}\log\log\frac{x}{p}}\Big)\right\}\leqslant$$

$$\leqslant\Lambda\Big(\frac{\beta u_{l+1}}{u_{l+1}+1}\Big)\frac{c_{qy}x}{\log^2 x}\Big(\log\frac{u_{l+1}}{u_l}+\frac{u_{l+1}-u_l}{u_{l+1}u_l}\Big)+O\Big(\frac{c_{qy}x}{\log^3 x}\Big)+$$

$$+O\Big(\frac{c_{qy}x}{\log^2 x\log\log x}\Big(\log\frac{u_{l+1}}{u_l}+\frac{u_{l+1}-u_l}{u_{l+1}u_l}\Big)\Big).$$

由於當 $x\geqslant 1$ 時, $\dfrac{x}{1+x}$ 是 x 的遞增函數. 故

$$\sum_{l=0}^{n-1}\Lambda\Big(\frac{\beta u_{l+1}}{u_{l+1}+1}\Big)\Big\{\log\frac{u_{l+1}}{u_l}+\frac{u_{l+1}-u_l}{u_{l+1}u_l}\Big\}-\int_{\alpha-1}^{\beta-1}\Lambda\Big(\frac{\beta z}{z+1}\Big)\frac{z+1}{z^2}dz=$$

$$=\sum_{l=0}^{n-1}\int_{u_l}^{u_{l+1}}\Big(\Lambda\Big(\frac{\beta u_{l+1}}{u_{l+1}+1}\Big)-\Lambda\Big(\frac{\beta z}{z+1}\Big)\Big)\frac{z+1}{z^2}dz\leqslant$$

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$$\leqslant \sum_{l=0}^{n-1}\left(\Lambda\left(\frac{\beta u_{l+1}}{u_{l+1}+1}\right)-\Lambda\left(\frac{\beta u_l}{u_l+1}\right)\right)\max_{0\leqslant l\leqslant n-1}\int_{u_l}^{u_{l+1}}\frac{z+1}{z^2}dz=$$

$$=O\left(\frac{1}{n}\right).$$

故

$$\sum_{\substack{x^{\frac{1}{\beta}}<p\leqslant x^{\frac{1}{\alpha}}\\ p\nmid y}} P_{\omega_p}\left(\frac{x}{p}, q, x^{\frac{1}{\beta}}\right)=\sum_{l=0}^{n-1}T_l\leqslant\left(\int_{\alpha-1}^{\beta-1}\Lambda\left(\frac{\beta z}{z+1}\right)\frac{z+1}{z^2}dz\right)\frac{c_{qy}x}{\log^2 x}+$$

$$+O\left(\frac{c_{qy}x}{\log^2 x\log\log x}\right).$$

定理證完.

定理 B$_2$. 若 $2\leqslant\alpha<\beta\leqslant 15$ 是固定兩數，又若有非負遞增且僅有有限多個不連續點之函數 $\Lambda(z)$ 及 $\lambda(z)$ $(0<z\leqslant 15)$ 使

$$P_\omega(x, q, x^{\frac{1}{z}})\geqslant\lambda(z)\frac{c_{qy}x}{\log^2 x}+O\left(\frac{c_{qy}x}{\log^2 x\log\log x}\right)\quad(0<z\leqslant 15)$$

及

$$P_\omega(x, q, x^{\frac{1}{z}})\leqslant\Lambda(z)\frac{c_{qy}x}{\log^2 x}+O\left(\frac{c_{qy}x}{\log^2 x\log\log x}\right)\quad(0<z\leqslant 15),$$

此處與“O”有關之常數爲絕對常數，則

$$\lambda_1(\alpha)=\begin{cases}0, & 2<\alpha\leqslant\tau;\\ \lambda(\beta)-2\int_{\alpha-1}^{\beta-1}\Lambda(z)\frac{z+1}{z^2}dz, & \tau\leqslant\alpha\leqslant\beta\leqslant 15\end{cases}$$

與

$$\Lambda_1(\alpha)=\Lambda(\beta)-2\int_{\alpha-1}^{\beta-1}\lambda(z)\frac{z+1}{z^2}dz,\quad 2\leqslant\alpha\leqslant\beta\leqslant 15$$

亦分別具有 $\lambda(\alpha)$ 與 $\Lambda(\alpha)$ 之性質.

證明見 Бухштаб[9].

6. 定理 C

給予兩數 $2\leqslant y\leqslant x$. 給出一組數

(ω) $$a, q; a_i, b_i\quad(i=1, 2, \cdots)$$

滿足

(12) $2\,|\,q$, $q=O(1)$; 若 $p_i|y$, 則 $a_i\equiv b_i\ (\mathrm{mod}\ p_i)$, 否則 $a_i\not\equiv b_i\ (\mathrm{mod}\ p_i)$, $(i=1, 2, \cdots)$,

此處 $2<p_1<p_2<\cdots$ 爲不能整除 q 的全體素數.

命 u, v 爲滿足 $15\geqslant v>u>1$ 之兩數. 以 $\mathfrak{M}$ 表示適合下面條件的整數 n 的集合:

(13) $1\leqslant n\leqslant x$, $n\equiv a\ (\mathrm{mod}\ q)$, $n\not\equiv a_i\ (\mathrm{mod}\ p_i)$, $n\not\equiv b_i\ (\mathrm{mod}\ p_i)$, $(1\leqslant i\leqslant s)$, $n\not\equiv a_{s+j}\ (\mathrm{mod}\ p_{s+j}^2)$, $n\not\equiv b_{s+j}\ (\mathrm{mod}\ p_{s+j}^2)$, $(1\leqslant j\leqslant t-s)$,

此處 $p_s\leqslant x^{\frac{1}{v}}<p_{s+1}$, $p_t\leqslant x^{\frac{1}{u}}<p_{t+1}$. $\mathfrak{M}$ 中元素的個數記之以 $M(x, x^{\frac{1}{v}}, x^{\frac{1}{u}})$.

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本段的宗旨在於證明

定理 C. 最多滿足下面 $t-s$ 個關係式：

(14) $$n \equiv c_{s+j} \pmod{p_{s+j}}, \ (1 \leqslant j \leqslant t-s, \ c_{s+j} \text{ 爲 } a_{s+j} \text{ 或 } b_{s+j})$$

中之 m 個的 $\mathfrak{M}$ 的元素個數不少於

$$P_\omega(x, q, x^{\frac{1}{v}}) - \left(\frac{2}{m+1}\int_{u-1}^{v-1} \Lambda\left(\frac{vz}{z+1}\right)\frac{z+1}{z^2}dz\right)\frac{c_{qy}x}{\log^2 x} + O\left(\frac{c_{qy}x}{\log^2 x \log\log x}\right).$$

證明之前，先證次之二引：

引 7. $M(x, x^{\frac{1}{v}}, x^{\frac{1}{u}}) = P_\omega(x, q, x^{\frac{1}{v}}) + O(x^{\frac{1}{u}}) + O(x^{1-\frac{1}{v}})$.

證. $P_\omega(x, q, x^{\frac{1}{v}}) - M(x, x^{\frac{1}{v}}, x^{\frac{1}{u}}) \leqslant$

$$\leqslant \sum_{j \leqslant t-s}\Big\{\sum_{\substack{n \leqslant x \\ n \equiv a_{s+j} \pmod{p_{s+j}^2}}} 1 + \sum_{\substack{n \leqslant x \\ n \equiv b_{s+j} \pmod{p_{s+j}^2}}} 1\Big\} \leqslant$$

$$\leqslant \sum_{j \leqslant t-s}\left(2\left[\frac{x}{p_{s+j}^2}\right] + 2\right) =$$

$$= O\Big(\sum_{n > x^{\frac{1}{v}}} \frac{1}{n^2}\Big) + O\Big(\sum_{n \leqslant x^{\frac{1}{u}}} 1\Big) = O(x^{1-\frac{1}{v}}) + O(x^{\frac{1}{u}}),$$

引理成立.

引 8. 存在 (ω_j) 及 $(\tilde{\omega}_j)$ $(1 \leqslant j \leqslant t-s)$ 使 $\mathfrak{M}$ 中的元素，它至少適合關係式 (14) 中的 l 個者之個數不超過

$$\frac{1}{l}\Big\{\sum_{\substack{j \leqslant t-s \\ p_{s+j} \nmid y}}\left(P_{\omega_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right) + P_{\tilde{\omega}_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)\right)\Big\} + O(x^{1-\frac{1}{v}}).$$

證. 當 $1 \leqslant j \leqslant t-s$ 時，$\mathfrak{M}$ 中之元素之具有性質

$$n \equiv c_{s+j} \pmod{p_{s+j}} \quad (c_{s+j} \text{ 爲 } a_{s+j} \text{ 或 } b_{s+j})$$

者之全體，吾人以 Γ_j 記之.

(i) $p_{s+j} \mid y$: 由假定 $a_{s+j} \equiv b_{s+j} \pmod{p_{s+j}}$. 當 $1 \leqslant i \leqslant s$ 時，下面同餘式

$$\begin{cases} a_{s+j} + mp_{s+j} \equiv a \pmod q & (1 \leqslant m \leqslant q), \\ a_{s+j} + mp_{s+j} \equiv a_i \pmod{p_i} & (1 \leqslant m \leqslant p_i), \\ a_{s+j} + mp_{s+j} \equiv b_i \pmod{p_i} & (1 \leqslant m \leqslant p_i) \end{cases}$$

均在所示區間中有唯一的解. 分別記之爲 $a^{(j)}, a_i^{(j)}, b_i^{(j)}$. 顯然當 $p_i \mid y$ 時，$a_i^{(j)} = b_i^{(j)}$，否則 $a_i^{(j)} \neq b_i^{(j)}$. 命

(ω_j) 　　$a^{(j)}, q; a_i^{(j)}, b_i^{(j)} \quad (i = 1, 2, \cdots, s)$.

則 Γ_j 的元素個數顯然不超過 $P_{\omega_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)$.

(ii) $p_{s+j} \nmid y$: 由假定 $a_{s+j} \not\equiv b_{s+j} \pmod{p_{s+j}}$. 當 $1 \leqslant i \leqslant s$ 時，同樣記下面同餘式之解分別爲 $\tilde{a}^{(j)}, \tilde{a}_i^{(j)}, \tilde{b}_i^{(j)}$.

$$\begin{cases} b_{s+j} + mp_{s+j} \equiv a \pmod q & (1 \leqslant m \leqslant q), \\ b_{s+j} + mp_{s+j} \equiv a_i \pmod{p_i} & (1 \leqslant m \leqslant p_i), \\ b_{s+j} + mp_{s+j} \equiv b_i \pmod{p_i} & (1 \leqslant m \leqslant p_i). \end{cases}$$

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命

$$(\tilde{\omega}_j)\qquad \tilde{a}^{(j)},\ q;\ \tilde{a}_i^{(j)},\ \tilde{b}_i^{(j)}\quad (i=1,2,\cdots,s),$$

則 Γ_j 的元素個數顯然不超過 $P_{\omega_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)+P_{\tilde{\omega}_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)$.

又 $P_{\omega_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)=O\left(\sum_{n\leqslant x/p_{s+j}} 1\right)=O(x^{1-\frac{1}{v}})$.

若 $n\in\mathfrak{M}$, 它至少滿足(14)中的 l 個, 則 $\mathfrak{M}$ 至少屬於 l 個不同的 Γ_j. 故至少適合關係式(14)中的 l 個的 $\mathfrak{M}$ 的元素個數不超過

$$\frac{1}{l}\left\{\sum_{\substack{j\leqslant t-s\\ p_{s+j}\nmid y}}\left(P_{\omega_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)+P_{\tilde{\omega}_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)\right)+\sum_{\substack{j\leqslant t-s\\ p_{s+j}\mid y}}P_{\omega_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)\right\}=$$

$$=\frac{1}{l}\sum_{\substack{j\leqslant t-s\\ p_{s+j}\nmid y}}\left(P_{\omega_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)+P_{\tilde{\omega}_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)\right)+O(x^{1-\frac{1}{v}}).$$

引理證完.

定理 C 的證明. 由引 7, 引 8 及定理 B_1 可知最多滿足(14)中 m 個的 $\mathfrak{M}$ 的元素個數不少於

$$M(x, x^{\frac{1}{v}}, x^{\frac{1}{u}})-\frac{1}{m+1}\left(\sum_{\substack{j\leqslant t-s\\ p_{s+j}\nmid y}}\left(P_{\omega_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)+\right.\right.$$

$$\left.\left.+P_{\tilde{\omega}_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)\right)\right)+O(x^{1-\frac{1}{v}})=$$

$$=P_{\omega}(x, q, x^{\frac{1}{v}})-\frac{1}{m+1}\sum_{\substack{j\leqslant t-s\\ p_{s+j}\nmid y}}\left(P_{\omega_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)+\right.$$

$$\left.+P_{\tilde{\omega}_j}\left(\frac{x}{p_{s+j}}, q, x^{\frac{1}{v}}\right)\right)+O(x^{1-\frac{1}{v}})+O(x^{\frac{1}{u}})\geqslant$$

$$\geqslant P_{\omega}(x, q, x^{\frac{1}{v}})-\left(\frac{2}{m+1}\int_{u-1}^{v-1}\Lambda\left(\frac{vz}{z+1}\right)\frac{z+1}{z^2}\,dz\right)\frac{c_{qy}x}{\log^2 x}+$$

$$+O\left(\frac{c_{qy}x}{\log^2 x\log\log x}\right)$$

定理證完.

7. 主要定理的證明

命 $\lambda(\alpha)$ 及 $\Lambda(\alpha)$ $(0<\alpha\leqslant 15)$ 爲使下式成立的, 僅有有限多個不連續點的非負遞增函數:

$$\lambda(\alpha)\frac{c_{qy}x}{\log^2 x}+O\left(\frac{c_{qy}x}{\log^2 x\log\log x}\right)\leqslant P_{\omega}(x, q, x^{\frac{1}{\alpha}})\leqslant \Lambda(\alpha)\frac{c_{qy}x}{\log^2 x}+ \tag{15}$$

$$+O\left(\frac{c_{qy}x}{\log^2 x\log\log x}\right)\quad (0<\alpha\leqslant 15),$$

此處與"O"有關的常數爲絕對常數.

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這種函數記之以 $\lambda_0(\alpha)$, $\Lambda_0(\alpha)$; $\lambda_1(\alpha)$, $\Lambda_1(\alpha)$; $\cdots$.

命 $x_i = 3.5 + 0.01i (0 \leqslant i \leqslant 210)$, $x_{210+i} = 5.6 + 0.1i\ (1 \leqslant i \leqslant 34)$. 用(6), (8), (10)(取 ε 足够小)直接算出 $\Lambda(x_i)(0 \leqslant i \leqslant 232)$. 再用 Бухштаб[10] 中的 $\lambda(10)$ 及 $\Lambda(10)$ 及 [9] [10] 所提出的方法以求出 $\Lambda(x_i)$ $(233 \leqslant i \leqslant 244)$ 及 $\lambda(x_i)$ $(0 \leqslant i \leqslant 244)$. 當 $x_i < x \leqslant x_{i+1}$ 時, 定義 $\Lambda(x) = \Lambda(x_{i+1})$. 當 $x_i \leqslant x < x_{i+1}$ 時, 定義 $\lambda(x) = \lambda(x_i)$. 則如此構造出來的函數具有(15)之性質. 記之爲 $\lambda_0(x)$ 及 $\Lambda_0(x)$. 現在將其在整點所取之值寫於下:

(16)

α	10	9	8	7	6	5	4
$\lambda_0(\alpha)$	99.98181*	79.78469	60.88817	43.51554	26.70925	9.18109	0
$\Lambda_0(\alpha)$	100.02073*	82.7207	68.52511	54.39352	43.0082	34.89666	29.39023

將區間 $\alpha - 1 < x \leqslant \beta - 1$ 分爲 n 個小區間 $u_i < x \leqslant u_{i+1}$ $(0 \leqslant i \leqslant n-1)$, $u_0 = \alpha - 1$, $u_n = \beta - 1$. 則由於 $\lambda(\alpha)$ 及 $\Lambda(\alpha)$ 均爲遞增函數, 故

$$\int_{\alpha-1}^{\beta-1} \lambda(z)\frac{z+1}{z^2}dz \geqslant \sum_{s=0}^{n-1}\lambda(u_s)\int_{u_s}^{u_{s+1}}\frac{z+1}{z^2}dz,$$

$$\int_{\alpha-1}^{\beta-1} \Lambda(z)\frac{z+1}{z^2}dz \leqslant \sum_{s=0}^{n-1}\Lambda(u_{s+1})\int_{u_s}^{u_{s+1}}\frac{z+1}{z^2}dz.$$

取 $u_{i+1} - u_i = 0.01$. 利用此二式及定理 B_2, 從 $\lambda_0(\alpha)$ 及 $\Lambda_0(\alpha)$ 出發, 經過幾次計算, 得到

(17)

α	10	9	8	7	6	……	$0<\alpha<5.53$
$\lambda_{11}(\alpha)$	99.98181	80.892035	63.59931	47.471252	31.004145	……	…………
$\Lambda_{12}(\alpha)$	100.02073	81.11841	64.403149	50.529826	41.01897	……	$\Lambda_{12}(\alpha)=\Lambda_0(\alpha)$

I. 取 $x = y$ 爲偶數, 又取

(ω_1)　　$a = 1,\ q = 2;\ a_i = 0,\ b_i = x \quad (i = 1, 2, \cdots)$,

在此 p_i 爲第 i 個奇素數.

(i) 在定理 C 內取 $v = 8$, $u = 2$, $m = 3$, 則由表(16)可知

$$P_{\omega_1}(x, 2, x^{\frac{1}{8}}) - \left(\frac{1}{2}\int_1^7 \Lambda_0\left(\frac{8z}{z+1}\right)\frac{z+1}{z^2}dz\right)\frac{c_{2x}x}{\log^2 x} + O\left(\frac{c_{2x}x}{\log^2 x \log\log x}\right) >$$

$$> \Big\{60.88817 - \frac{1}{2}\Big[\Lambda_0(7)\int_4^7\frac{z+1}{z^2}dz + \Lambda_0(6.4)\int_3^4\frac{z+1}{z^2}dz +$$

$$+ \Lambda_0(6)\int_2^3\frac{z+1}{z^2}dz + \Lambda_0(5.4)\int_{1.28}^2\frac{z+1}{z^2}dz +$$

$$+ \Lambda_0(4.5)\int_1^{1.28}\frac{z+1}{z^2}dz\Big]\Big\}^{**}\frac{c_{2x}x}{\log^2 x} + O\left(\frac{c_{2x}x}{\log^2 x\log\log x}\right) >$$

$$> 0.56125\frac{c_{2x}x}{\log^2 x} + O\left(\frac{c_{2x}x}{\log^2 x\log\log x}\right) > 3 \quad (x > x_0).$$

* $\Lambda(10) = 100.02073$ 及 $\lambda(10) = 99.98191$ 取自 Бухштаб[10].

** 此處用到下面簡單的事實: 若 $f(x)$ 與 $g(x)$ 是區間 $a \leqslant x \leqslant b$ 中的非負函數, 且 $f(x)$ 爲遞增的, 則當 $a < c - \delta$, $c < b$ 時,

$$\int_a^b g(x)f(x)dx < f(b)\int_{c-\delta}^b g(x)dx + f(c)\int_a^{c-\delta} g(x)dx.$$

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故由定理C得知，當 $x>x_0$ 時，存在整數 n，滿足 $1<n<x-1$，$n(x-n)$ 不能被 $\leqslant x^{\frac{1}{8}}$ 的素數整除，最多被區間 $x^{\frac{1}{8}}<p\leqslant x^{\frac{1}{2}}$ 中3個素數整除，但不被此區間的素數平方整除，其他的素因子均大於 $x^{\frac{1}{2}}$. 故 $n(x-n)$ 爲不超過5個素數的乘積. 由於 $x=n+x-n$. 故得 (a, b) $(a+b\leqslant 5)$.

(ii) 取 $v=6$, $u=3$, $m=2$, 則由表(16)可知

$$P_{\omega_1}(x, 2, x^{\frac{1}{6}})-\left(\frac{2}{3}\int_2^5 \Lambda_0\left(\frac{6z}{z+1}\right)\frac{z+1}{z^2}dz\right)\frac{c_{2x}x}{\log^2 x}+O\left(\frac{c_{2x}x}{\log^2 x\log\log x}\right)>$$

$$>\Bigg\{26.70925-\frac{2}{3}\bigg[\Lambda_0(5)\int_{\frac{4.9}{1.1}}^5\frac{z+1}{z^2}dz+\Lambda_0(4.9)\int_4^{\frac{4.9}{1.1}}\frac{z+1}{z^2}dz+$$

$$+\Lambda_0(4.8)\int_{\frac{4.6}{1.4}}^4\frac{z+1}{z^2}dz+\Lambda_0(4.6)\int_{\frac{4.4}{1.6}}^{\frac{4.6}{1.4}}\frac{z+1}{z^2}dz+$$

$$+\Lambda_0(4.4)\int_{\frac{4.2}{1.8}}^{\frac{4.4}{1.6}}\frac{z+1}{z^2}dz+\Lambda_0(4.2)\int_2^{\frac{4.2}{1.8}}\frac{z+1}{z^2}dz\bigg]\Bigg\}\frac{c_{2x}x}{\log^2 x}+$$

$$+O\left(\frac{c_{2x}x}{\log^2 x\log\log x}\right)>3\quad(x>x_0).$$

同上可知 $(3, 3)$.

(iii) 取 $v=8$, $u=\frac{16}{7}$, $m=2$, 則由表(17)可知

$$P_{\omega_1}(x, 2, x^{\frac{1}{8}})-\left(\frac{2}{3}\int_{\frac{9}{7}}^7 \Lambda_{12}\left(\frac{8z}{z+1}\right)\frac{z+1}{z^2}dz\right)\frac{c_{2x}x}{\log^2 x}+O\left(\frac{c_{2x}x}{\log^2 x\log\log x}\right)>$$

$$>\left(\lambda_{11}(8)-\frac{128}{3}\sum_{y=0}^{124}\frac{\Lambda_{12}(4.5+0.02\cdot y)}{(4.5+0.02\cdot y)^2}\int_{\frac{4.5+0.02\cdot y}{3.5-0.02\cdot y}}^{\frac{4.5+0.02\cdot(y+1)}{3.5-0.02\cdot(y+1)}}\frac{dz}{z+1}\right)^{*}\frac{c_{2x}x}{\log^2 x}+$$

$$+O\left(\frac{c_{2x}x}{\log^2 x\log\log x}\right)>$$

$$>0.43\frac{c_{2x}x}{\log^2 x}+O\left(\frac{c_{2x}x}{\log^2 x\log\log x}\right)>3\quad(x>x_0).$$

同上可知 $(2, 3)$.

II. 取 x 爲一整數，$y=k$ 爲一固定偶數. 又取

$$(\omega_2)\qquad a=1,\ q=2;\ a_i=0,\ b_i=-k\quad(i=1, 2, \cdots),$$

此處 p_i 爲第 i 個奇素數. 在定理C內取 $v=8$, $u=\frac{16}{7}$, $m=2$, 則由(17)得

$$P_{\omega_2}(x, 2, x^{\frac{1}{8}})-\left(\frac{2}{3}\int_{\frac{9}{7}}^7 \Lambda_{12}\left(\frac{8z}{z+1}\right)\frac{z+1}{z^2}dz\right)\frac{c_{2k}x}{\log^2 x}+O\left(\frac{c_{2k}x}{\log^2 x\log\log x}\right)>$$

$$>0.43\frac{c_{2k}x}{\log^2 x}+O\left(\frac{c_{2k}x}{\log^2 x\log\log x}\right)>0.4\frac{c_{2k}x}{\log^2 x}\quad(x>x_0),$$

---

* 此處用到了函數 $\frac{\Lambda_{12}(z)}{z^2}$ 的遞減性，當 $5.53\leqslant z\leqslant 10$ 時，由[10]可知 $\frac{\Lambda_{12}(z)}{z^2}$ 遞減，當 $0<z<5.53$ 時，由(17)可知.

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故由定理C得知當 $x > x_0$ 時,區間 $1 \leqslant n \leqslant x$ 中多於 $\frac{0.4c_{2k}x}{\log^2 x}$ 個 n 使 $n(n+k)$ 不被 $\leqslant x^{\frac{1}{8}}$ 的素數整除，最多被區間 $x^{\frac{1}{8}} < p \leqslant x^{\frac{7}{16}}$ 中2個素數整除，但又不被該區間中任何素數的平方整除，其餘的素因子均大於 $x^{\frac{7}{16}}$. 因此 $n(n+k)$ 爲不超過5個素數的乘積，且 n 與 $n+k$ 的素因子個數均不多於3. 故得定理2.

III. 取 $x = y$ 爲奇數,又取

$$(\omega_3)\qquad a = x-2,\ q = 4;\ a_i = 0,\ b_i = x\quad (i = 1, 2, \cdots),$$

此處 p_i 爲第 i 個奇素數. 在定理C內取 $v = 8$, $u = \frac{16}{7}$, $m = 2$, 則由(17)得

$$P_{\omega_3}(x, 4, x^{\frac{1}{8}}) - \left(\frac{2}{3}\int_{\frac{9}{7}}^{7} \Lambda_{12}\left(\frac{8z}{z+1}\right)\frac{z+1}{z^2}dz\right)\frac{c_{4x}x}{\log^2 x} + O\left(\frac{c_{4x}x}{\log^2 x \log\log x}\right) >$$
$$> 3\quad (x > x_0).$$

故由定理C得知,當 $x > x_0$ 時,區間 $1 < n < x-1$ 存在 n 滿足: $\frac{n(x-n)}{2}$ 爲一整數, $\frac{n(x-n)}{2}$ 不能被 $\leqslant x^{\frac{1}{8}}$ 的素數整除，最多被區間 $x^{\frac{1}{8}} < p \leqslant x^{\frac{7}{16}}$ 中二個素數整除，其餘的素因子均大於 $x^{\frac{7}{16}}$. 故得定理3.

附錄：關於 Selberg 方法的若干附記

一切記號如前幾段所示,不一一說明. 以 ν_x 表示素因子皆 $< x$ 之無平方因子數.

定理 A′. 命 $c \geqslant 1$, $P = \prod_{\substack{p \leqslant \xi \\ p \nmid q}} p$. 則對於任意給予的整數列 (ω), 下式一致成立

$$P_\omega(x, q, \xi) \geqslant \frac{x}{q}\left(1 - \sum_{p \mid P} g(p)\frac{1}{\sum\limits_{\substack{\nu_p \leqslant \xi^c/\sqrt{p} \\ \nu_p \mid P}} \frac{\mu^2(\nu_p)}{f(\nu_p)}}\right) + O(\xi^{2c}\log^7 \xi),$$

此處 $g(n)$ 與 $f(n)$ 之定義一如定理A.

證明參看 А. И. Виноградов[3].

當 $z \leqslant \xi$ 時,命

$$(18)\qquad \sum_{\substack{\nu_z \leqslant \xi^c/\sqrt{z} \\ \nu_z \setminus P}} \frac{\mu^2(\nu_z)}{f(\nu_z)} = \prod_{\substack{p < z \\ p \nmid q}}\left(1 + \frac{1}{f(p)}\right)(1 - \varepsilon(u_z)) = \frac{1 - \varepsilon(u_z)}{\Pi_z},$$

此處 $\Pi_z = \prod_{\substack{p < z \\ p \nmid q}} (1 - g(p))$, $u_z = \frac{1}{2}\left(2c\frac{\log z}{\log x} - 1\right)$.

А. И. Виноградов[3] 曾給 $\varepsilon(u_z)$ 一個解析表示式.

取 $\xi = \left(\frac{x^{\frac{1}{2}}}{\log^5 x}\right)^{\frac{1}{c}}$. 由 Mertens 定理得

$$\Pi_\xi = qc_{qy}\frac{1}{\log^2 \xi}\left(1 + O\left(\frac{1}{\log x}\right)\right),$$

故得

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$$\lim_{\xi\to\infty}\sum_{\substack{\nu\leqslant\xi^c\\ \nu|P}}\frac{\mu^2(\nu)}{f(\nu)}\cdot\Pi_\xi=\frac{(2c)^2}{\Lambda(2c)},$$

此處 $\Lambda(2c)$ 由公式 (6), (8), (10) 等所定義. 故由(18)得

$$1-\varepsilon(c)=\frac{(2c)^2}{\Lambda(2c)}\quad\left(c\geqslant\frac{1}{2}\right), \tag{19}$$

此式即 $\varepsilon(c)$ 與 $\Lambda(d)$ 之間的關係. 由於 $\varepsilon(c)$ 遞減, 故 $\dfrac{\Lambda(d)}{d^2}$ 亦遞減.

不難算出(參看[3])

$$P_\omega(x,q,\xi)\geqslant\frac{x}{q}\prod_{p|P}\left(1-g(p)-\sum_{p|P}g(p)\Pi_p\frac{\varepsilon(u_p)}{1-\varepsilon(u_p)}\right)+O\left(\frac{x}{\log^3 x}\right)=$$

$$=\frac{xc_{qy}}{\log^2 x}\left(4c^2-4\int_{\frac{2c-1}{2}}^{\lambda}\frac{\varepsilon(u)}{1-\varepsilon(u)}(2u+1)du\right)+O\left(\frac{x}{\log^{2.5}x}\right).$$

因此得出

$$\lambda(d)=\max\left(0,\ d^2-4\int_{\frac{d-1}{2}}^{\lambda}\frac{\varepsilon(u)}{1-\varepsilon(u)}(2u+1)du\right). \tag{20}$$

通過實際計算可知

$$(4.1)^2-4\int_{1.55}^{\lambda}\frac{\varepsilon(u)}{1-\varepsilon(u)}(2u+1)du<16.81-4\int_{1.55}^{3}\frac{\varepsilon(u)}{1-\varepsilon(u)}(2u+1)\,du<$$

$$<16.81-4\sum_{i=1}^{14}\frac{\varepsilon(1.5+0.1x(i+1))}{1-\varepsilon(1.5+0.1x(i+1))}\int_{1.5+0.1xi}^{1.5+0.1x(i+1)}(2u+1)du-$$

$$-4\frac{\varepsilon(1.6)}{1-\varepsilon(1.6)}\int_{1.55}^{1.6}(2u+1)\,du<-1.$$

故僅(當) $d>4.1$ 時,(20)式才有用.

參 考 文 獻

[1] 王元: 論篩法及其有關的若干問題. 科學記錄,新輯 1 卷 1 期 (1957), 9—11.

[2] 王元: 表大偶數爲兩個殆素數之和. 科學記錄,新輯 1 卷 5 期 (1957), 15—18.

[3] А. И. Виноградов, Применение $\zeta(s)$ к решету эратосфена, *Матем. сб.*, **41**: 1 (1957), 49—80.

[4] А. И. Виноградов, Письмо в редакцию, *Матем. сб.*, **41**: 3 (1957), 415—416.

[5] A. Selberg, The general sieve method and its place in prime number theory, *Proc. of the Inter. Cong. of Math.*, **1** (1950), 286—292.

[6] H. N. Shapiro and J. Warga, On representation of large integers as sum of primes, part I, *Comm. pure Appl. Math.*, **3** (1950), 153—176.

[7] А. А. Бухштаб, Асиптотическая оценка одной общей теоретикочисловой функции, *Матем. сб.*, **2** (1937), 1239—1245.

[8] 王元: 表大偶數爲一個不超過三個素數的乘積及一個不超過四個素數的乘積之和. 數學學報, **3** (1956), 500—513.

[9] А. А. Бухштаб, Новые улучшения в методе эратосфенова решета, *Матем. сб.*, **4** (1938), 375—387.

[10] А. А. Бухштаб, О разложении чётных чисел на сумму двух слагаемых с ограниченным числом множителей, ДАН СССР, **29** (1940), 544—548.

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ON SIEVE METHODS AND SOME OF THEIR APPLICATIONS (I)

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ABSTRACT

In this paper, we give the details of the proofs of the following three theorems (Cf. Science Record, Academia Sinica, I: 1 (1957), 9—12; I: 3 (1957), 1—5).

Theorem 1. Every sufficiently large even integer can be written as a sum of two positive numbers > 1, of which one contains at most 2 and other at most 3 prime factors.

Theorem 2. For any given even number k, there are infinitely many integers n, such that each of n and $n+k$ has at most 3 prime factors and $n(n+k)$ is a product of not more than 5 primes.

Theorem 3. Every sufficiently large odd integer can be represented as $2N+1=2P+Q \quad (P>1, Q>1)$, where the number of prime factors of P and also of Q is not more than 3 and PQ is a product of at most 5 primes.

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