GLOBAL UNIQUE SOLVABILITY OF INHOMOGENEOUS NAVIER-STOKES EQUATIONS WITH BOUNDED DENSITY

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ABSTRACT. In this paper, we prove the global existence and uniqueness of solution to d-dimensional (for $d = 2, 3$) incompressible inhomogeneous Navier-Stokes equations with initial density being bounded from above and below by some positive constants, and with initial velocity $u_0 \in H^s(\mathbb{R}^2)$ for $s > 0$ in 2-D, or $u_0 \in H^1(\mathbb{R}^3)$ satisfying $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2}$ being sufficiently small in 3-D. This in particular improves the most recent well-posedness result in [10], which requires the initial velocity $u_0 \in H^2(\mathbb{R}^d)$ for the local well-posedness result, and a smallness condition on the fluctuation of the initial density for the global well-posedness result.

Keywords: Inhomogeneous Navier-Stokes equations, well-posedness, Lagrangian coordinates.

AMS Subject Classification (2000): 35Q30, 76D05

1. Introduction

In this paper, we consider the global existence and uniqueness of the solution to the following $d$-dimensional (for $d = 2, 3$) incompressible inhomogeneous Navier-Stokes equations with initial density in $L^\infty(\mathbb{R}^d)$ and having a positive lower bound:

$$
\begin{aligned}
\partial_t \rho + u \cdot \nabla \rho &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
\rho(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla p &= 0, \\
\text{div} u &= 0, \\
(\rho, u)|_{t=0} &= (\rho_0, u_0),
\end{aligned}
$$

(1.1)

where $\rho, u$ stand for the density and velocity of the fluid respectively, $p$ is a scalar pressure function, and the viscosity coefficient is supposed to be 1. Such a system describes a fluid which is obtained by mixing two miscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. One may check [17] for the detailed derivation of this system.

Given $0 \leq \rho_0 \in L^\infty(\mathbb{R}^d)$, and $u_0$ satisfying $\text{div} u_0 = 0$, $\sqrt{\rho_0} u_0 \in L^2(\mathbb{R}^d)$, Lions [17] (see also [5, 19] and the references therein for an overview of results on weak solutions of (1.1)) proved that (1.1) has a global weak solution so that

$$
\frac{1}{2} \|\sqrt{\rho(t)} u(t)\|^2_{L^2} + \int_0^t \|\nabla u(\tau)\|^2_{L^2} \, d\tau \leq \frac{1}{2} \|\sqrt{\rho_0} u_0\|^2_{L^2}.
$$

Moreover, for any $\alpha$ and $\beta$, the Lebesgue measure

$$
\mu\{x \in \mathbb{R}^d; \quad \alpha \leq \rho(t,x) \leq \beta \} \quad \text{is independent of} \quad t.
$$

In dimension two and under the additional assumption that $\rho_0$ is bounded below by a positive constant and $\nabla u_0 \in L^2(\mathbb{R}^2)$, smoother weak solutions may be built. Their existence stems from a quasi-conservation law involving the norm of $\nabla u \in L^\infty((0,T);L^2(\mathbb{R}^2))$ and of $\partial_t u, \nabla p$ and $\nabla^2 u \in L^2((0,T);L^2(\mathbb{R}^2))$ for any $T < \infty$. For both types of weak solutions however, the problem of uniqueness has not been solved.

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Ladyženskaja and Solonnikov [16] first addressed the question of unique solvability of (1.1). More precisely, they considered the system (1.1) in a bounded domain $\Omega$ with homogeneous Dirichlet boundary condition for $u$. Under the assumption that $u_0 \in W^{2-\frac{2}{p},p}(\Omega)$ ($p > d$) is divergence free and vanishes on $\partial \Omega$ and that $\rho_0 \in C^1(\Omega)$ is bounded away from zero, then they [16] proved

- Global well-posedness in dimension $d = 2$;
- Local well-posedness in dimension $d = 3$. If in addition $u_0$ is small in $W^{2-\frac{2}{p},p}(\Omega)$, then global well-posedness holds true.

More recently, Danchin [8] established the well-posedness of the system (1.1) in the whole space $\mathbb{R}^d$ for small perturbations of some constant density. Abidi, Gui and Zhang [3] investigated the large time decay and global stability to any global smooth solutions of (1.1).

Another important feature of (1.1) is the scaling invariant property: if $(\rho, u)$ is a solution of (1.1) associated to the initial data $(\rho_0, u_0)$, then $(\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x))$ is also a solution of (1.1) associated to the initial data $(\rho_0(\lambda x), \lambda u_0(\lambda x))$. A functional space for the data $(\rho_0, u_0)$ or for the solution $(\rho, u)$ is said to be at the scaling of the equation if its norm is invariant under the above transformation. In this framework, it has been stated in [1, 7] that for the initial data $(\rho_0, u_0)$ satisfying

$$(\rho_0 - 1) \in \dot{B}^{\frac{d}{p},1}_{p,1}(\mathbb{R}^d), \ u_0 \in \dot{B}^{\frac{d-1}{p},1}_{p,1}(\mathbb{R}^d)$$

with $\text{div} \ u_0 = 0$

and that for a small enough constant $c$

$$\|\rho_0 - 1\|_{\dot{B}^{\frac{d}{p},1}_{p,1}} + \|u_0\|_{\dot{B}^{\frac{d-1}{p},1}_{p,1}} \leq c,$$

we have for any $p \in [1, 2d]$

- existence of global solution $(\rho, u, \nabla p)$ with $\rho - 1 \in C_b([0, \infty); \dot{B}^{\frac{d}{p},1}_{p,1}(\mathbb{R}^d))$, $u \in C_b([0, \infty); \dot{B}^{\frac{d-1}{p},1}_{p,1}(\mathbb{R}^d))$;
- $\partial_t u, \nabla^2 u, \nabla p \in L^1(\mathbb{R}^d; \dot{B}^{\frac{d-1}{p},1}_{p,1}(\mathbb{R}^d));$
- uniqueness in the above space if in addition $p \leq d$.

These results have been somewhat extended in [2] so that $u_0$ belongs to a larger Besov space. Paicu and Zhang [18] further extended the well-posedness result in [2] so that even if one component of the initial velocity is large, (1.1) still has a unique global solution. The smallness assumption for the initial density in [1, 7] has also been removed in [4], and the restriction of $p \in [1, d]$ for uniqueness result in [1, 7] has been removed recently in [9].

A byproduct in [4] implies the global existence of solutions to (1.1) in 3-D with initial density in $L^\infty(\mathbb{R}^3)$ and having a positive lower bound, and initial velocity being sufficiently small in $H^2(\mathbb{R}^3)$. The authors [9] proved the global well-posedness of (1.1) provided that

$$\|\rho_0 - 1\|_{\mathcal{M}(\dot{B}^{\frac{d-1}{p},1}_{p,1})} + \|u_0\|_{\dot{B}^{\frac{d-1}{p},1}_{p,1}} \leq c,$$

for some sufficiently small constant $c$, where $\mathcal{M}(\dot{B}^{\frac{d-1}{p},1}_{p,1}(\mathbb{R}^d))$ denotes the multiplier space of $\dot{B}^{\frac{d-1}{p},1}_{p,1}(\mathbb{R}^d)$. This space in particular includes initial densities having small jumps across a $C^1$ interface.

Again in the scaling invariant framework, the authors [13] proved the global existence of weak solutions to (1.1) provided that the initial data satisfy the nonlinear smallness condition:

$$(\|\rho_0^{-1} - 1\|_{L^\infty} + \|u_0^h\|_{\dot{B}^{\frac{d-1}{p},\#}_{p,r}} \exp(C_r\|u_0^d\|_{\dot{B}^{\frac{d-1}{p},\#}}^{2r}) \leq c_0\mu)$$

for some positive constants $c_0, C_r$ and $1 < p < d$, $1 < r < \infty$, where $u_0^h = (u_0^1, \cdots, u_0^{d-1})$ and $u_0 = (u_0^h, u_0^d)$. With a little bit more regularity assumption on the initial velocity, they [13] also proved the uniqueness of such solutions.
In general when \( \rho_0 \in L^\infty(\mathbb{R}^d) \) with a positive lower bound and \( u_0 \in H^2(\mathbb{R}^d) \), Danchin and Mucha [10] proved that the system (1.1) has a unique local solution. Furthermore, with the initial density fluctuation being sufficiently small, for any initial velocity \( u_0 \in B^1_{1,2}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \) in two space dimensions, and \( u_0 \in B^{2 - \frac{2}{d}}_{q,p}(\mathbb{R}^d) \) with \( 1 < p < \infty, d < q < \infty \) and \( 2 - \frac{2}{p} \neq \frac{1}{q} \), they also proved the global well-posedness of (1.1).

On the other hand, Hoff [14, 15] proved the global existence of small energy solutions to the isentropic compressible Navier-Stokes system. The main idea in [14, 15] is that with appropriate time weight (see Remark 1.2 for details), one can close the energy estimate for space derivatives of the velocity field even if the initial velocity only belongs to \( L^2(\mathbb{R}^d) \). Motivated by [14, 15], we shall investigate the global well-posedness of (1.1) with less regular initial velocity than that in [10] and without the small fluctuation assumption on the initial density. We emphasize that the Lagrangian idea introduced in [9, 10] will also be essential for the proof of the uniqueness result here.

Our main results in this paper can be listed as follows.

**Theorem 1.1.** Let \( s > 0 \). Given the initial data \((\rho_0, u_0)\) satisfying

\[
0 < c_0 \leq \rho_0(x) \leq C_0 < +\infty, \quad u_0 \in H^3(\mathbb{R}^2),
\]

the system (1.1) has a unique global solution \((\rho, u)\) such that

\[
c_0 \leq \rho(t,x) \leq C_0 \quad \text{for} \quad (t, x) \in [0, +\infty) \times \mathbb{R}^2,
\]

\[
a_0(t) \leq C\|u_0\|_{L^2}^2,
\]

\[
a_1(t) \leq C\|u_0\|_{H^s}^2 \exp\{C\|u_0\|_{L^2}^4\},
\]

\[
a_2(t) \leq C(1 + \|u_0\|_{L^2}^8(\|u_0\|_{H^s}^2 + \|u_0\|_{H^s}^4)) \exp\{C\|u_0\|_{L^2}^4\},
\]

for any \( t \in [0, +\infty) \). Here \( C \) is a constant depending on \( c_0, C_0 \), and \( a_0(t), a_1(t), \) and \( a_2(t) \) are defined by

\[
a_0(T) \overset{\text{def}}{=} \frac{1}{2} \sup_{t \in [0,T]} \int_{\mathbb{R}^2} \rho|u(t,x)|^2 \, dx + \int_0^T \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \, dt,
\]

\[
a_1(T) \overset{\text{def}}{=} \frac{1}{2} \sup_{t \in [0,T]} \sigma(t)^{1-s} \int_{\mathbb{R}^2} |\nabla u(t,x)|^2 \, dx + \int_0^T \int_{\mathbb{R}^2} \sigma(t)^{1-s}(\rho|u_t|^2 + |\nabla^2 u| + |\nabla p|^2) \, dx \, dt,
\]

\[
a_2(T) \overset{\text{def}}{=} \frac{1}{2} \sup_{t \in [0,T]} \sigma(t)^{2-s} \int_{\mathbb{R}^2} (\rho|u_t(t,x)|^2 + |\nabla^2 u(t,x)|^2 + |\nabla p|^2) \, dx
\]

\[\quad + \int_0^T \int_{\mathbb{R}^2} \sigma(t)^{2-s}|\nabla u_t|^2 \, dx \, dt,\]

with \( \sigma(t) \overset{\text{def}}{=} \min(1,t) \).

**Theorem 1.2.** Given the initial data \((\rho_0, u_0)\) satisfying

\[
0 < c_0 \leq \rho_0(x) \leq C_0 < +\infty, \quad u_0 \in H^1(\mathbb{R}^3),
\]

there exists a constant \( \varepsilon_0 > 0 \) depending only on \( C_0 \) such that if

\[
\|u_0\|_{L^2} \|\nabla u_0\|_{L^2} \leq \varepsilon_0,
\]

the system (1.1) has a unique global solution \((\rho, u)\) which satisfies

\[
c_0 \leq \rho(t,x) \leq C_0 \quad \text{for} \quad (t, x) \in [0, +\infty) \times \mathbb{R}^3,
\]

\[
B_0(t) \leq \|\rho_0 u_0\|_{L^2}^\frac{2}{q},
\]

\[
B_1(t) \leq 2\|\nabla u_0\|_{L^2}^2,
\]

\[
B_2(t) \leq C(1 + \|\nabla u_0\|_{L^2}^4) \|\nabla u_0\|_{L^2}^2 \exp\{C\|u_0\|_{L^2}^4 + \|\nabla u_0\|_{L^2}^2\},
\]

where \( \rho_0 \overline{\leq} \rho_0 \) and \( u_0 \overline{\leq} u_0 \).
for any $t \in [0, +\infty)$. Here $C$ is a constant depending on $C_0$, and $B_0(t), B_1(t),$ and $B_2(t)$ are defined by

$$B_0(T) \equiv \sup_{t \in [0, T]} \int_{\mathbb{R}^3} \rho|u(t, x)|^2 \, dx + 2 \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \, dx dt,$$

$$B_1(T) \equiv \sup_{t \in [0, T]} \int_{\mathbb{R}^3} |\nabla u(t, x)|^2 \, dx + 2 \int_0^T \int_{\mathbb{R}^3} (\rho|u|^2 + |\nabla^2 u|^2 + |\nabla p|^2) \, dx dt,$$

$$B_2(T) \equiv \sup_{t \in [0, T]} \sigma(t) \int_{\mathbb{R}^3} (\rho|u(t, x)|^2 + |\nabla^2 u(t, x)|^2 + |\nabla p|^2) \, dx + \int_0^T \int_{\mathbb{R}^3} \sigma(t)|\nabla u|^2 \, dx dt.$$

**Remark 1.1.** We should point out that we do not need the lower bound assumption for the initial density in the existence part of Theorem 1.2. Indeed, the constant $C$ in (1.6) is independent of $c_0$ in (1.4). We can also prove the local existence and uniqueness solution to the system (1.1) even if the initial velocity does not satisfy the smallness condition (1.5). One may check Remark 2.1 for details.

**Remark 1.2.** The powers to the weight $\sigma(t)$ in the energy functionals, $A_i(t), B_i(t)$ for $i = 1, 2$, are motivated by the following observation: let $s$ be a negative real number and $(p, r) \in [1, \infty]^2$, a constant $C$ exists such that

$$C^{-1} \|f\|_{B^s_{p, r}} \leq \|t^{-\frac{s}{2}} e^{t \Delta} f\|_{L^r(\mathbb{R}^d)} \|f\|_{B^s_{p, r}} \leq C \|f\|_{B^s_{p, r}}^r,$$

(see Theorem 2.34 of [6] for instance). In particular, if $u_0 \in H^s(\mathbb{R}^2)$ for $s \in (0, 1), \nabla u_0 \in H^{s-1}(\mathbb{R}^2) \hookrightarrow B^{2s-1}_{2, 2}(\mathbb{R}^2)$ and $\nabla^2 u_0 \in B^{2s-2}_{2, \infty}(\mathbb{R}^2)$. Then according to (1.7),

$$t^{\frac{1-s}{2}} \|e^{t \Delta} \nabla u_0\|_{L^2} \leq C \|u_0\|_{H^s} \quad \text{and} \quad t^{2s-2} \|e^{t \Delta} \nabla^2 u_0\|_{L^2} \leq C \|u_0\|_{H^s}.$$ 

This in some sense explains the weights in (1.3). Similarly, we can explain the weights in (1.6).

**Remark 1.3.** We should point out that we can not directly apply Theorem 1 of [10] concerning the uniqueness of solutions to (1.1) to conclude the uniqueness part of Theorem 1.1 and Theorem 1.2. Yet the Lagrangian idea in [9, 10] can be successfully applied to prove the uniqueness part of both Theorems 1.1 and 1.2. And the uniqueness result of Germain [11] can not be applied here either. The uniqueness result of [11] requires the density function satisfying $\nabla \rho \in L^\infty([0, T]; L^d(\mathbb{R}^d))$, but here our density function only belongs to $L^\infty([0, T] \times \mathbb{R}^d)$. Moreover, the velocity field in Theorems 1.1 and 1.2 does not satisfy the time growth condition in [11], especially in Theorem 1.1.

2. Global solutions to (1.1) with large bounded density

The purpose of this section is to present the proof to the existence part of both Theorem 1.1 and Theorem 1.2.

2.1. Existence of the solution in 2-D

**Proof to the existence part of Theorem 1.1.** Let $j_\epsilon$ be the standard Friedrich’s mollifier. We define

$$\rho^\epsilon_0 = j_\epsilon \ast \rho_0, \quad u^\epsilon_0 = j_\epsilon \ast u_0.$$ 

And we choose $\epsilon$ so small that

$$\frac{c_0}{2} \leq \rho^\epsilon_0(x) \leq 2C_0, \quad x \in \mathbb{R}^2.$$ 

With the initial data $(\rho^\epsilon_0, u^\epsilon_0)$, the system (1.1) in 2-D has a unique global smooth solution $(\rho^\epsilon, u^\epsilon)$. In what follows, we shall only present uniform energy estimates (1.3) for the approximate solutions $(\rho^\epsilon, u^\epsilon)$. Then the existence part of Theorem 1.1 essentially follows from (1.3) for $(\rho^\epsilon, u^\epsilon)$ and a standard compactness argument. The uniqueness part of Theorem 1.1 will be proved in Section 3.
To simplify the notations, we will omit the superscript $\epsilon$ in what follows. First of all, applying the basic $L^2$ energy estimate to (1.1) gives

$$A_0(t) = \frac{1}{2} \|\rho_0^\frac{1}{2} u_0\|_{L^2}^2 \leq C \|u_0\|_{L^2}^2 \quad \text{for} \quad t \in \mathbb{R}^+.$$  

While it follows from the transport equation of (1.1) and (1.2) that

$$c_0 \leq \rho(t, x) \leq C_0 \quad \text{for} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2.$$  

To derive the estimate for $A_1(t)$, we get, by taking the $L^2$ inner product of the momentum equation of (1.1) with $u_t$, that

$$\int_{\mathbb{R}^2} \rho |u_t|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx = - \int_{\mathbb{R}^2} \rho (u \cdot \nabla u) \cdot u_t \, dx,$$

from which, we infer

$$\tilde{A}_1(t) \overset{\text{def}}{=} \int_0^t \int_{\mathbb{R}^2} \sigma(\tau) \rho |u_t|^2 \, dx \, d\tau + \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^2} |\nabla u(t, x)|^2 \, dx$$

$$\leq - \int_0^t \int_{\mathbb{R}^2} \sigma(\tau) \rho (u \cdot \nabla u) \cdot u_t \, dx \, d\tau + \int_0^t \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \, d\tau.$$

In this subsection, we shall frequently use the following version of Gagliardo-Nirenberg inequality:

$$\|a\|_{L^p(\mathbb{R}^2)} \leq C \|a\|_{L^2(\mathbb{R}^2)}^\frac{2}{p} \|\nabla a\|_{L^2(\mathbb{R}^2)}^{1-\frac{2}{p}} \quad \text{for} \quad 2 \leq p < \infty.$$  

By virtue of (2.2) and (2.4), we obtain

$$\int_0^t \int_{\mathbb{R}^2} \sigma(\tau) \rho (u \cdot \nabla u) \cdot u_t \, dx \, d\tau$$

$$\leq C \int_0^t \sigma(\tau) \|u(\tau)\|_{L^4} \|\nabla u(\tau)\|_{L^4} \|\rho^\frac{1}{2} u_t(\tau)\|_{L^2} \, d\tau$$

$$\leq C_\delta \int_0^t \sigma(\tau) \|u(\tau)\|_{L^4}^2 \|\nabla u(\tau)\|_{L^4}^2 \|\Delta u(\tau)\|_{L^2} \, d\tau + \frac{1}{2} \int_0^t \sigma(\tau) \|\rho^\frac{1}{2} u_t(\tau)\|_{L^2}^2 \, d\tau$$

$$\leq C_\delta A_0(t) \int_0^t \sigma(\tau) \|\nabla u(\tau)\|_{L^2}^4 \, d\tau + \frac{1}{2} \int_0^t \sigma(\tau) \|\Delta u(\tau)\|_{L^2}^2 \, d\tau + \delta \tilde{A}_1(t),$$

for any $\delta > 0$, where $C_\delta$ is a positive constant so that $C_\delta \to \infty$ as $\delta \to 0$. Whereas thanks to (1.1), we write

$$-\Delta u + \nabla p = -\rho (u_t + u \cdot \nabla u),$$

which along with the classical estimate on the Stokes system ensures that

$$\|\nabla^2 u\|_{L^2} + \|\nabla p\|_{L^2} \leq C (\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2})$$

$$\leq C (\|\rho u_t\|_{L^2} + \|u\|_{L^2}^2 \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}}),$$

so that

$$\|\nabla^2 u\|_{L^2} + \|\nabla p\|_{L^2} \leq C (\|\rho^\frac{1}{2} u_t\|_{L^2} + \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^{\frac{1}{2}}).$$

Substituting (2.7) into (2.5) gives rise to

$$\int_0^t \int_{\mathbb{R}^2} \sigma(\tau) \rho (u \cdot \nabla u) \cdot u_t \, dx \, d\tau \leq C_\delta A_0(t) \int_0^t \sigma(\tau) \|\nabla u(\tau)\|_{L^2}^4 \, d\tau + C \delta \tilde{A}_1(t).$$
We define the linear operator 
\[ T v \]
Taking the to the momentum equation of (1.1), that 
\[ T v \]
Applying Gronwall’s inequality gives 
\[ (2.9) \]
To obtain the estimate of \( A_1(t) \), we need to use an interpolation argument. For this, we consider the linear momentum equation
\[ \rho(\partial_t v + u \cdot \nabla v) - \Delta v + \nabla p = 0, \quad v(0, x) = v_0(x). \]

Then it follows from the same line to the proof of (2.9) and (2.7) that 
\[ \int_0^t \int_{\mathbb{R}^2} (\rho|v_t|^2 + |\nabla v|^2 + |\nabla p|^2) \, dx \, dt \leq C \| v_0 \|_{L^2}^2 \exp \{ C \| u_0 \|_{L^2}^4 \}; \]
\[ \int_0^t \int_{\mathbb{R}^2} \sigma(\tau)(\rho|v_t|^2 + |\nabla v|^2 + |\nabla p|^2) \, dx \, dt + \sigma(t) \int_0^t \int_{\mathbb{R}^2} |\nabla v(t, x)|^2 \, dx \leq C \| v_0 \|_{L^2}^2 \exp \{ C \| u_0 \|_{L^2}^4 \}. \]

We define the linear operator \( T v_0 = \nabla v \). The above inequalities tell us that
\[ \| T v_0 \|_{L^2} \leq C \exp \{ C \| u_0 \|_{L^2}^4 \} \| v_0 \|_{H^1}, \]
\[ \| T v_0 \|_{L^2} \leq C \sigma(t)^{-\frac{1}{12}} \exp \{ C \| u_0 \|_{L^2}^4 \} \| v_0 \|_{L^2}, \]
from which and Riesz-Thorin interpolation theorem [12], we infer
\[ \| \nabla v \|_{L^2} = \| T v_0 \|_{L^2} \leq C \sigma(t)^{-\frac{1}{12}} \exp \{ C \| u_0 \|_{L^2}^4 \} \| v_0 \|_{H^s}. \]

Further, we define a family of operators \( T_2 v_0 = \sigma(t)^{\frac{s}{2}} \rho^{\frac{s}{2}} \partial_t v(t, x) \) for \( \Re s \in [0, 1] \). Then we have
\[ \| T_{iy} v_0 \|_{L^2(0, t; L^2)} \leq C \exp \{ C \| u_0 \|_{L^2}^4 \} \| v_0 \|_{H^1}, \]
\[ \| T_{i+y} v_0 \|_{L^2(0, t; L^2)} \leq C \exp \{ C \| u_0 \|_{L^2}^4 \} \| v_0 \|_{L^2}, \]
for any \( y \in \mathbb{R} \). Apply Stein interpolation theorem [12] to get
\[ \| \sigma(t)^{-\frac{s}{2}} \rho^{s} \partial_t v \|_{L^2(0, t)} = \| T_{1-s} v_0 \|_{L^2(0, t; L^2)} \leq C \exp \{ C \| u_0 \|_{L^2}^4 \} \| v_0 \|_{H^s}. \]

The other terms can be treated in a similar way. Therefore we have
\[ (2.10) \quad A_1(t) \leq C \| u_0 \|_{H^s}^2 \exp \{ C \| u_0 \|_{L^2}^4 \}. \]

Finally, we manipulate the \( H^2 \) energy estimate for \( u \). We first get, by taking the time derivative to the momentum equation of (1.1), that
\[ \rho(u_t + u \cdot \nabla u) - \Delta u_t + \nabla p_t = -\rho_t(u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u. \]
Taking the \( L^2 \) inner product of the above equation with \( u_t \), and then using integration by parts, we write
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \rho|u_t|^2 \, dx + \int_{\mathbb{R}^2} |\nabla u_t|^2 \, dx \]
\[ = -\int_{\mathbb{R}^2} \rho_t |u_t|^2 \, dx - \int_{\mathbb{R}^2} \rho_t (u_t \cdot \nabla u) \cdot u_t \, dx - \int_{\mathbb{R}^2} \rho (u_t \cdot \nabla u) \cdot u_t \, dx, \]
from which, we infer
\[
\tilde{A}_2(t) \overset{\text{def}}{=} \frac{1}{2} \sigma(t)^{2-s} \int_{\mathbb{R}^2} \rho |u_t|^2 \, dx + \int_0^t \sigma(\tau)^{2-s} \int_{\mathbb{R}^2} |\nabla u_t|^2 \, dx \, d\tau
\]
\[
= - \int_0^t \sigma(\tau)^{2-s} \int_{\mathbb{R}^2} \rho_t |u_t|^2 \, dx \, d\tau - \int_0^t \sigma(\tau)^{2-s} \int_{\mathbb{R}^2} \rho_t (u \cdot \nabla u) \cdot u_t \, dx \, d\tau
\]
\[
- \int_0^t \sigma(\tau)^{2-s} \int_{\mathbb{R}^2} \rho (u_t \cdot \nabla u) \cdot u_t \, dx \, d\tau + (2 - s) \int_0^t \sigma(\tau)^{1-s} \int_{\mathbb{R}^2} \rho |u_t|^2 \, dx \, d\tau
\]
\[
\overset{\text{def}}{=} A + B + C + D.
\]

It is obvious to check that
\[
|D| \leq (2 - s)A_1(t),
\]
and
\[
C \leq \int_0^t \sigma(\tau)^{2-s} \|\rho\|_{L^\infty} \|\nabla u\|_{L^2} \|u_t\|_{L^4}^2 \, d\tau
\]
\[
\leq C \int_0^t \sigma(\tau)^{2-s} \|\nabla u\|_{L^2} \|u_t\|_{L^2}^2 \|\nabla u_t\|_{L^2} \, d\tau
\]
\[
\leq C \int_0^t \sigma(\tau)^{2-s} \|\nabla u\|_{L^2}^2 \|u_t\|_{L^2}^2 \|\rho\|^2 \frac{1}{2} |u_t|_{L^2}^2 \, d\tau + \frac{1}{4} \tilde{A}_2(t).
\]

While noticing that $\rho_t = -u \cdot \nabla \rho$, we get by using integration by parts and (2.4) that
\[
A = -2 \int_0^t \sigma(\tau)^{2-s} \int_{\mathbb{R}^2} \rho u_t \cdot (u \cdot \nabla u_t) \, dx \, d\tau
\]
\[
\leq 2 \int_0^t \sigma(\tau)^{2-s} \|u_t\|_{L^4} \|\rho\|_{L^\infty} \|\nabla u_t\|_{L^2} \|u_t\|_{L^4} \, d\tau
\]
\[
\leq C \int_0^t \sigma(\tau)^{2-s} \|u_t\|_{L^2}^2 \|\nabla u\|_{L^2} \|u_t\|_{L^2}^2 \|\nabla u_t\|_{L^2} \, d\tau
\]
\[
\leq C \int_0^t \sigma(\tau)^{2-s} \|u_t\|_{L^2}^2 \|\nabla u\|_{L^2} \|u_t\|_{L^2}^2 \, d\tau + \frac{1}{4} \int_0^t \sigma(\tau)^{2-s} \|\nabla u_t\|_{L^2}^2 \, d\tau
\]
\[
\leq C \|u_0\|_{L^2}^2 \int_0^t \sigma(\tau)^{2-s} \|\nabla u\|_{L^2} \|u_t\|_{L^2}^2 \, d\tau + \frac{1}{4} \tilde{A}_2(t).
\]

Along the same line, we write $B$ as
\[
B = \int_0^t \sigma(\tau)^{2-s} \int_{\mathbb{R}^2} \rho (u \cdot \nabla u) \cdot (u \cdot \nabla u_t) \, dx \, d\tau
\]
\[
+ \int_0^t \sigma(\tau)^{2-s} \int_{\mathbb{R}^2} \rho ((u \cdot \nabla u) \cdot \nabla u) \cdot u_t \, dx \, d\tau
\]
\[
+ \int_0^t \sigma(\tau)^{2-s} \int_{\mathbb{R}^2} \rho ((u \otimes u) : \nabla^2) u \cdot u_t \, dx \, d\tau
\]
\[
\overset{\text{def}}{=} B_1 + B_2 + B_3.
\]
By virtue of Hölder inequality and (2.4), one has
\[
B_1 \leq \int_0^t \sigma(\tau)^{2-s} \|ho\|_{L^\infty} \|u\|_{L^s}^2 \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} d\tau \\
\leq C \int_0^t \sigma(\tau)^{2-s} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \|\nabla u_t\|_{L^2} d\tau \\
\leq C \int_0^t \sigma(\tau)^{2-s} \|u\|_{L^2} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} d\tau + \frac{1}{16} \tilde{A}_2(t),
\]
which along with (2.7) implies
\[
B_1 \leq C \left\{ \int_0^t \sigma(\tau)^{2-s} \|\nabla u\|_{L^2}^2 \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 d\tau \\
+ \int_0^t \sigma(\tau)^{2-s} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \|\nabla u_t\|_{L^2}^2 d\tau \right\} + \frac{1}{16} \tilde{A}_2(t) \\
\leq C \int_0^t \sigma(\tau)^{2-s} \|\nabla u\|_{L^2}^2 \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 d\tau + C \|u_0\|_{L^2}^4 A_1(t)^2 + \frac{1}{16} \tilde{A}_2(t).
\]
The same argument gives rise to
\[
B_2 \leq \int_0^t \sigma(\tau)^{2-s} \|\rho\|_{L^\infty} \|u\|_{L^4} \|\nabla u\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2} d\tau \\
\leq C \int_0^t \sigma(\tau)^{2-s} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2} d\tau \\
\leq C \int_0^t \sigma(\tau)^{2-s} \|u\|_{L^2} \|\nabla u\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 d\tau + \int_0^t \sigma(\tau)^{2-s} \|\Delta u\|_{L^2}^3 d\tau \\
\leq C \int_0^t \sigma(\tau)^{2-s} \|u\|_{L^2} \|\nabla u\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 d\tau + C \int_0^t \sigma(\tau)^{2-s} \|\rho^{\frac{1}{2}} u_t\|_{L^2}^3 + \|u\|_{L^2} \|\nabla u\|_{L^2}^6 d\tau \\
\leq C \left( \|u_0\|_{L^2} \int_0^t \sigma(\tau)^{2-s} \|\nabla u\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 d\tau + A_1(t)^2 + \|u_0\|_{L^2}^2 A_1(t)^2 \right) + \frac{1}{16} \tilde{A}_2(t),
\]
and
\[
B_3 \leq \int_0^t \sigma(\tau)^{2-s} \|\rho\|_{L^\infty}^{\frac{1}{2}} \|u\|_{L^\infty} \|\nabla^2 u\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2} d\tau \\
\leq C \int_0^t \sigma(\tau)^{2-s} \|u\|_{L^2} \|\Delta u\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2} d\tau \\
\leq C \int_0^t \sigma(\tau)^{2-s} \|u\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 d\tau + \int_0^t \sigma(\tau)^{2-s} \|u\|_{L^2} \|\nabla u\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 d\tau \\
\leq C \left( \|u_0\|_{L^2}^2 A_1(t)^2 + \|u_0\|_{L^2}^2 A_1(t)^2 + \int_0^t \sigma(\tau)^{2-s} \|\nabla u\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 d\tau \right) + \frac{1}{16} \tilde{A}_2(t).
\]
Summing up the above estimates, we conclude that
\[
B \leq C \left( 1 + \|u_0\|_{L^2}^8 \right) \int_0^t \sigma(\tau)^{2-s} \|\nabla u\|_{L^2}^2 \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 d\tau \\
+ C \left( 1 + \|u_0\|_{L^2}^8 \right) A_1(t)^2 + \frac{3}{16} \tilde{A}_2(t).
\]
(2.15)
Combining (2.11) with (2.12)-(2.15), we obtain
\[
\bar{A}_2(t) \leq C \left\{ (1 + \|u_0\|_{L^2}^2) \int_0^t \sigma(\tau)^{2-s} \|\nabla u\|_{L^2}^2 \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 \, d\tau \\
+ (1 + \|u_0\|_{L^2}^8) A_1^2(t) + A_1(t) \right\}.
\]
Applying Gronwall's inequality and (2.10) leads to
\[
\bar{A}_2(t) \leq C \left( 1 + \|u_0\|_{L^2}^8 \right) \left( \|u_0\|_{H^s}^2 + \|u_0\|_{H^s}^4 \right) \exp \{ C \|u_0\|_{L^2}^4 \},
\]
which together with (2.7) ensures that
\[
A_2(t) \leq C (\bar{A}_2(t) + \|u_0\|_{L^2}^2 A_1^2(t)) \\
\leq C \left( 1 + \|u_0\|_{L^2}^8 \right) \left( \|u_0\|_{H^s}^2 + \|u_0\|_{H^s}^4 \right) \exp \{ C \|u_0\|_{L^2}^4 \},
\]
This together with (2.1) and (2.10) completes the proof of (1.3).

2.2. Existence of the solution in 3-D.

Proof to the existence part of Theorem 1.2. By mollifying the initial density $\rho_0$, we deduce from [4] that (1.1) has a unique global solution $(\rho^\varepsilon, u^\varepsilon)$ provided that $\varepsilon_0$ is small enough in (1.5). Then the existence part of Theorem 1.2 follows from the uniform estimate (1.6) for $(\rho^\varepsilon, u^\varepsilon)$ and a standard compactness argument. For simplicity, we only present the a priori estimates (1.6) for smooth enough solutions $(\rho, u)$ of (1.1). The uniqueness of such solution will be proved in Section 3. As a convention in the rest of this section, we shall always denote by $C$ a constant depending on $C_0$ in (1.4), which may be different from line to line.

First of all, it is easy to check from (1.1) and (1.4) that
\[
c_0 \leq \rho(t, x) \leq C_0 \quad \text{for} \quad (t, x) \in [0, +\infty) \times \mathbb{R}^3,
\]
(2.16)
\[
\int_{\mathbb{R}^3} \rho |u(t, x)|^2 \, dx + 2 \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt = \int_{\mathbb{R}^3} \rho_0 |u_0|^2 \, dx.
\]

While we get by taking the $L^2$ inner product of the momentum equation to (1.1) and $u_t$ that
\[
2 \int_0^t \int_{\mathbb{R}^3} \rho |u_t|^2 \, dx \, dt + \int_{\mathbb{R}^3} |\nabla u(t, x)|^2 \, dx \leq \|\nabla u_0\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} \rho (u \cdot \nabla u) \cdot u_t \, dx \, dt.
\]
(2.17)

In what follows, we need to use Gagliardo-Nirenberg inequality
\[
\|a\|_{L^p(\mathbb{R}^3)} \leq C \|a\|_{L^2(\mathbb{R}^3)}^{\frac{2}{p}} \|\nabla a\|_{L^2(\mathbb{R}^3)}^{\frac{2}{p}} \quad \text{for} \quad 2 \leq p \leq 6.
\]
(2.18)

By virtue of (2.18), one has
\[
\int_0^t \int_{\mathbb{R}^3} \rho (u \cdot \nabla u) \cdot u_t \, dx \, dt \leq \int_0^t \|\rho\|_{L^\infty} \|u\|_{L^6} \|\nabla u\|_{L^3} \|\rho^{\frac{1}{2}} u_t\|_{L^2} \, d\tau
\leq C \int_0^t \|\nabla u\|_{L^2}^{\frac{3}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} \|\rho^{\frac{1}{2}} u_t\|_{L^2} \, d\tau
\leq C \int_0^t \|\nabla u\|_{L^2}^3 \|\Delta u\|_{L^2} \, d\tau + \frac{1}{4} \int_0^t \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 \, d\tau.
\]

Whereas it follows from the momentum equation of (1.1) and classical estimates on the Stokes system that
\[
\|\nabla^2 u\|_{L^2} + \|\nabla p\|_{L^2} \leq C \left( \|\rho^{\frac{1}{2}} u_t\|_{L^2} + \|u \cdot \nabla u\|_{L^2} \right)
\leq C \left( \|\rho^{\frac{1}{2}} u_t\|_{L^2} + \|\nabla u\|_{L^2}^3 \right) + \frac{1}{2} \|\nabla^2 u\|_{L^2},
\]
(2.19)
so that
\[ \int_0^t \int_{\mathbb{R}^3} \rho (u \cdot \nabla u) \cdot u_t \, dx \, d\tau \leq C \int_0^t \| \nabla u \|_{L^2}^6 \, d\tau \quad \text{and} \quad \frac{1}{2} \int_0^t \| \rho^{\frac{3}{2}} u_t \|_{L^2}^2 \, d\tau, \]
from which, (2.17) and (2.19), we infer
\[ B_1(t) \leq \| \nabla u_0 \|_{L^2}^2 + C \| u_0 \|_{L^2}^2 B_1(t)^2. \]
Hence, as long as we choose \( \varepsilon_0 \) small enough in (1.6), we obtain the estimate for \( B_1(t) \) in (1.6).

We now turn to the estimate of \( B_2(t) \). Indeed along the same line to the proof of (2.11), we have
\[
\begin{align*}
& \frac{1}{2} \rho \| u_t \|_{L^2}^2 \, dx + \int_0^t \sigma(\tau) \int_{\mathbb{R}^3} |\nabla u_t|^2 \, dx \, d\tau \\
& \quad = - \int_0^t \sigma(\tau) \int_{\mathbb{R}^3} \rho |u_t|^2 \, dx \, d\tau - \int_0^t \sigma(\tau) \int_{\mathbb{R}^3} \rho_t (u \cdot \nabla u) \cdot u_t \, dx \, d\tau \\
& \quad \quad - \int_0^t \sigma(\tau) \int_{\mathbb{R}^3} \rho (u_t \cdot \nabla u) \cdot u_t \, dx \, d\tau + \int_0^t \int_{\mathbb{R}^3} \rho |u_t|^2 \, dx \, d\tau \\
& \quad \overset{\text{def}}{=} E + F + G + H.
\end{align*}
\]
It is obvious to observe that
\[ H \leq B_1(t) \leq 2 \| \nabla u_0 \|_{L^2}^2. \]
Whereas using \( \rho_t = -u \cdot \nabla \rho \) and integrating by parts, we get, by applying (2.18), that
\[
\begin{align*}
& E = -2 \int_0^t \sigma(\tau) \int_{\mathbb{R}^3} \rho (u \cdot \nabla u_t) \cdot u_t \, dx \, d\tau \\
& \leq 2 \int_0^t \sigma(\tau) \| u \|_{L^\infty} \| \rho \|_{L^\frac{4}{3}} \| \nabla u_t \|_{L^2} \| \rho^{\frac{1}{2}} u_t \|_{L^2} \, d\tau \\
& \leq C \int_0^t \sigma(\tau) \| \nabla u \|_{L^2}^\frac{1}{2} \| \nabla^2 u \|_{L^2}^\frac{1}{2} \| \nabla u_t \|_{L^2} \| \rho^{\frac{1}{2}} u_t \|_{L^2} \, d\tau \\
& \leq C \int_0^t \left( \| \nabla u \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2 \right) \sigma(\tau) \| \rho^{\frac{1}{2}} u_t \|_{L^2}^2 \, d\tau + \frac{1}{4} \int_0^t \sigma(\tau) \| \nabla u_t \|_{L^2}^2 \, d\tau.
\end{align*}
\]
Notice that
\[
\begin{align*}
& G \leq \int_0^t \sigma(\tau) \| \rho \|_{L^{\frac{4}{3}}} \| \nabla u \|_{L^3} \| u_t \|_{L^3} \| \rho^{\frac{1}{2}} u_t \|_{L^2} \, d\tau \\
& \leq C \int_0^t \sigma(\tau) \| \nabla u \|_{L^2}^\frac{1}{2} \| \nabla^2 u \|_{L^2}^\frac{1}{2} \| \nabla u_t \|_{L^2} \| \rho^{\frac{1}{2}} u_t \|_{L^2} \, d\tau.
\end{align*}
\]
(2.23) holds also for \( G \). To deal with \( F \) in (2.21), we get, by using \( \rho_t = -u \cdot \nabla \rho \) once again and integrating by parts, to write
\[
\begin{align*}
& F = \int_0^t \sigma(\tau) \int_{\mathbb{R}^3} \rho (u \cdot \nabla u) \cdot (u \cdot \nabla u_t) \, dx \, d\tau \\
& \quad + \int_0^t \sigma(\tau) \int_{\mathbb{R}^3} \rho (u \cdot \nabla u) \cdot u_t \, dx \, d\tau \\
& \quad + \int_0^t \sigma(\tau) \int_{\mathbb{R}^3} \rho (u \otimes u) : \nabla^2 u \cdot u_t \, dx \, d\tau \overset{\text{def}}{=} F_1 + F_2 + F_3.
\end{align*}
\]
By virtue of (1.6) for $B_1(t)$ and (2.18), one has

$$F_1 \leq \int_0^t \sigma(\tau) \|\rho\|_{L^\infty} \|u\|_{L^6}^2 \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2} \, d\tau$$

$$\leq C \int_0^t \sigma(\tau) \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} \|\nabla u_t\|_{L^2} \, d\tau$$

$$\leq C \int_0^t \sigma(\tau) \|\nabla u\|_{L^2}^4 \|\nabla^2 u\|_{L^2}^2 \, d\tau + \frac{1}{4} B_2(t) \leq C \|\nabla u_0\|_{L^2}^6 + \frac{1}{4} B_2(t).$$

Along the same line, we have

$$F_2 \leq \int_0^t \sigma(\tau) \|\rho\|_{L^\infty} \|u\|_{L^6} \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2} \, d\tau$$

$$\leq C \int_0^t \sigma(\tau) \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2} \, d\tau$$

$$\leq C \int_0^t \left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right)\sigma(\tau) \left(\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right) \, d\tau.$$

The same estimate holds for $F_3$, as

$$F_3 \leq \int_0^t \sigma(\tau) \|\rho\|_{L^\infty} \|u\|_{L^6}^2 \|\nabla u\|_{L^6} \|\nabla^2 u\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2} \, d\tau$$

$$\leq C \int_0^t \sigma(\tau) \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \|\rho^{\frac{1}{2}} u_t\|_{L^2} \, d\tau.$$

Therefore, we conclude that

$$F \leq C \int_0^t \left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right)\sigma(\tau) \left(\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right) \, d\tau + C \|\nabla u_0\|_{L^2}^6 + \frac{1}{4} B_2(t).$$

On the other hand, notice from (2.19) that

$$\sigma(t) \left(\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla p\|_{L^2}^2\right) \leq C \sigma(t) \left(\|\rho^{\frac{1}{2}} u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2\right) \leq C (B_2(t) + \|\nabla u_0\|_{L^2}^6),$$

which together with (2.21)-(2.24) ensures that

$$B_2(t) \leq C \left\{ \int_0^t \left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right) \sigma(\tau) \left(\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right) \, d\tau + \|\nabla u_0\|_{L^2}^6 + \|\nabla u_0\|_{L^2}^6 \right\},$$

applying Gronwall’s inequality gives rise to the estimate of $B_2(t)$ in (1.6). This completes the proof of (1.6). \qed

**Remark 2.1.** Along the same line to the derivation of (2.20), we also get

$$\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \int_{\mathbb{R}^3} \rho |u_t|^2 \, dx = -\int_{\mathbb{R}^3} \rho (u \cdot \nabla u) \cdot u_t \, dx$$

$$\leq C \|\nabla u\|_{L^2}^6 + \frac{1}{2} \int_{\mathbb{R}^3} \rho |u_t|^2 \, dx,$$

which gives

$$\frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \int_{\mathbb{R}^3} \rho |u_t|^2 \, dx \leq C \|\nabla u\|_{L^2}^6.$$

**Hence if the initial velocity $u_0$ does not satisfy (1.5), we deduce from (2.25) that there exists a positive time $\Delta$ so that**

$$\|\nabla u\|_{L^2(\Delta)}^2 + \|u_t\|_{L^2(\Delta)}^2 \leq C \|\nabla u_0\|_{L^2}^2.$$
With the above estimate, we can obtain the estimate $B_1(t)$ and $B_2(t)$ for $t \leq \tau$ as we did before. This implies the local existence of solutions to (1.1) in 3-D with large data.

3. Uniqueness of the Solution

3.1. More regularity of the solutions. Before we present the proof to the uniqueness part of both Theorem 1.1 and Theorem 1.2, we need the following regularity results for the solutions of (1.1) obtained in Section 2.

**Lemma 3.1.** Let $(\rho, u, \nabla p)$ be the global solution of (1.1) obtained in Theorem 1.2. Then for any $T \in \mathbb{R}^+$, one has

\[
\int_0^T \sigma(t) \left( \|\Delta u(t)\|_{L^6}^2 + \|\nabla p(t)\|_{L^6}^2 \right) dt \leq C,
\]

(3.1)

\[
\int_0^T \|\nabla u(t)\|_{L^\infty} dt \leq C \max(T^{\frac{1}{2}}, T^{\frac{1}{2}}),
\]

\[
\int_0^T \sigma(t)^{\frac{1}{2}} \|\nabla u(t)\|_{L^\infty} dt \leq C,
\]

for some constant $C$ depending only on $c_0, C_0$ in (1.4) and $\|u_0\|_{H^1}$.

**Proof.** We first get by taking $\text{div}$ to (2.6) that

\[
\nabla p = \nabla (-\Delta)^{-1} \text{div} \{\rho(\partial_t u + u \cdot \nabla u)\},
\]

which along with (1.6) and (2.6) ensures that

\[
\|\Delta u(t)\|_{L^q} + \|\nabla p(t)\|_{L^q} \leq C \|\partial_t u + u \cdot \nabla u(t)\|_{L^q} \quad \text{for any } q \in (1, \infty).
\]

However, it follows from Sobolev imbedding theorem that

\[
\|(u \cdot \nabla u)(t)\|_{L^6} \leq \|u(t)\|_{L^\infty} \|\Delta u(t)\|_{L^2} \leq C \|\nabla u(t)\|_{L^2}^\frac{1}{2} \|\Delta u(t)\|_{L^2}^\frac{3}{2},
\]

from which, (1.6) and (3.2), we infer that

\[
\begin{align*}
\int_0^T \sigma(t) \left( \|\Delta u(t)\|_{L^6}^2 + \|\nabla p(t)\|_{L^6}^2 \right) dt & \leq C \left\{ \int_0^T \sigma(t) \|\nabla u(t)\|_{L^2}^2 dt + \sup_{t \in [0, T]} \left( \|\nabla u(t)\|_{L^2} \sigma(t) \|\Delta u(t)\|_{L^2} \right) \int_0^T \|\Delta u(t)\|_{L^2}^2 dt \right\} \\
& \leq C.
\end{align*}
\]

This proves the first part of (3.1). Then we deduce from it, Gagliardo-Nirenberg inequality and (1.6) that

\[
\int_0^T \|\nabla u(t)\|_{L^\infty} dt \leq C \int_0^T \|\Delta u(t)\|_{L^6}^\frac{1}{2} \|\Delta u(t)\|_{L^6}^\frac{1}{2} dt \\
\leq \|\Delta u\|_{L^2(L^6)}^\frac{1}{2} \left( \int_0^T \sigma(t) \|\Delta u(t)\|_{L^6}^2 dt \right)^\frac{1}{2} \left( \int_0^T \sigma(t)^{-\frac{1}{2}} dt \right)^\frac{1}{2} \\
\leq C \left( \int_0^T \sigma(t)^{-\frac{1}{2}} dt \right)^\frac{1}{2} \leq C \max(T^{\frac{1}{2}}, T^{\frac{1}{2}}).
\]

Along the same line, we can also prove the estimate for $\int_0^T \sigma(t)^{\frac{1}{2}} \|\nabla u(t)\|_{L^\infty} dt$. \qed
The 2-D version of the above lemma is more complicated, which we present as follows.

**Lemma 3.2.** Let \((\rho, u, \nabla p)\) be the global solution of \((1.1)\) obtained in Theorem 1.1. Then for any \(T \in \mathbb{R}^+\) and \(\alpha \in [0,1),\) one has

\[
\int_0^T \sigma(t)^{1+\alpha-s} \left( \|\partial_t u, \Delta u(t)\|_{L^{\frac{2}{1+\alpha}}}^2 + \|\nabla p(t)\|_{L^{\frac{2}{1+\alpha}}}^2 \right) dt \leq C,
\]

(3.3)

\[
\int_0^T \|\nabla u(t)\|_{L^\infty} dt \leq C \max(T^{\frac{s}{2(1+\alpha)}}, T^\frac{1}{2}),
\]

\[
\int_0^T \sigma(t)^{1-\frac{1}{1+\alpha}} \|\nabla u(t)\|_{L^\infty}^2 dt \leq C,
\]

where the constant \(C\) depends on \(\alpha, c_0, C_0\) in (1.2) and \(\|u_0\|_{H^s}\).

**Proof.** First of all, for any \(\beta \in (0,1),\) we deduce from Gagliardo-Nirenberg inequality that

\[
\|a\|_{L^{\frac{2}{1+\alpha}}} \leq C \|a\|_{L^2}^{1-\beta} \|\nabla a\|_{L^2}^\beta \quad \text{for any} \quad a \in H^1(\mathbb{R}^2),
\]

which along with (1.3) ensures that for any \(\alpha, \beta, \gamma \in (0,1)\) satisfying \(\beta + \gamma = 1 + \alpha\)

\[
\begin{align*}
&\|\sigma(t)^{\frac{1}{2}+\alpha(1+\alpha)}\|((\rho u, \nabla u)(t))_{L^{\frac{2}{1+\alpha}}(T)} \leq C_0 \|\sigma(t)^{\frac{1}{2}+\alpha(1+\alpha)}\|u(t)_{L^{\frac{2}{1+\alpha}}(T)} \|\nabla u(t)\|_{L^{1+\gamma}}(T)_{L^2}, \\
&\leq C_0 \sup_{t \in [0,T]} \left( \|u(t)\|_{L^2}^{1-\beta} \sigma(t)^{\frac{1}{2}+\gamma} \|\nabla u(t)\|_{L^2}^{\beta} \right) \|\nabla u(t)\|_{L^2}^{1-\gamma} \sigma(t)^{\frac{1}{2}+\gamma} \|\nabla^2 u(t)\|_{L^2} \|L^2, \right.
\end{align*}
\]

(3.4)

Along the same line, we obtain the same estimate for \(\|\sigma(t)^{\frac{1}{2}+\alpha(1+\alpha)}\|\partial_t u\|_{L^{\frac{2}{1+\alpha}}(T)} \). On the other hand, it follows from (3.2) that

\[
\|\Delta u(t)\|_{L^{\frac{2}{1+\alpha}}} + \|\nabla p(t)\|_{L^{\frac{2}{1+\alpha}}} \leq \|(u_t + u \cdot \nabla u)(t)\|_{L^{\frac{2}{1+\alpha}}},
\]

from which and (3.4), we obtain the first inequality of (3.3).

Whereas we get, by using Gagliardo-Nirenberg inequality once again, that

\[
\begin{align*}
\int_0^T \|\nabla u(t)\|_{L^\infty} dt &\leq C \int_0^T \|\nabla u(t)\|_{L^{\frac{2}{1+\alpha}}} \|\Delta u(t)\|_{L^{\frac{2}{1+\alpha}}} dt \\
&\leq C \|\nabla u\|_{L^{\frac{2}{1+\alpha}}(L^2)} \|\sigma(t)^{\frac{1}{2}+\alpha(1+\alpha)}\|\Delta u(t)\|_{L^{\frac{2}{1+\alpha}}(L^2)} \left( \int_0^T \sigma(t)^{\frac{1}{2}+\alpha(1+\alpha)} dt \right)^{\frac{1}{2}},
\end{align*}
\]

which together with (1.3) ensures the second inequality of (3.3). Along the same line, we can prove the last inequality in (3.3). \(\square\)

3.2. **Lagrangian formulation.** As in [9, 10], we shall apply Lagrangian approach to prove the uniqueness of the solutions. We remark that even with (3.1) and (3.3), the solution of (1.1) obtained in Theorem 1.1 and Theorem 1.2 does not satisfy the assumptions required by Theorem 1 of [10] concerning the uniqueness of solutions to (1.1). Fortunately, the idea used to prove Theorem 1 of [10] can be successfully applied here.

Let \((\rho, u, p)\) be the solution of \((1.1)\) obtained in Theorem 1.1 and Theorem 1.2. Then thanks to (3.1) and (3.3), we can define the trajectory \(X(t, y)\) of \(u(t, x)\) by

\[\partial_t X(t, y) = u(t, X(t, y)), \quad X(0, y) = y.\]
which leads to the following relation between the Eulerian coordinates \( x \) and the Lagrangian coordinates \( y \):

\[
X(t, y) = y + \int_0^t u(\tau, X(\tau, y))d\tau.
\]

Moreover, we deduce from (3.1) and (3.3) that we can take \( T \) small enough such that

\[
\int_0^T \| \nabla u(t) \|_{L^\infty} dt \leq \frac{1}{2}.
\]

Then for \( t \leq T \), \( X(t, y) \) is invertible with respect to \( y \) variables, and we denote by \( Y(t, \cdot) \) its inverse mapping. Let \( v(t, y) \equiv u(t, x) = u(t, X(t, y)) \). One has

\[
\begin{align*}
\partial_t v(t, y) &= \partial_t u(t, x) + u(t, x) \cdot \nabla u(t, x), \\
\partial_x u^j(t, x) &= \partial_j v^i(t, y) \partial_x y^k \quad \text{for} \quad x = X(t, y), \quad y = Y(t, x).
\end{align*}
\]

Let \( A(t, y) \equiv (\nabla X(t, y))^{-1} = \nabla_x Y(t, x) \). So we have

\[
\nabla_x u(t, x) = A^t(t, x) \nabla_y v(t, y) \quad \text{and} \quad \text{div} u(t, x) = \text{div}(A(t, y)v(t, y)).
\]

By the chain rule, we also have

\[
\text{div}_y (A \cdot) = A^t \nabla_y.
\]

Here and in what follows, we always denote \( A^t \) the transpose matrix of \( A \).

As in [10], we denote

\[
\begin{align*}
\nabla u &= A^t \cdot \nabla_y, \quad \text{div}_u \equiv \text{div}(A \cdot) \quad \text{and} \quad \Delta_u \equiv \text{div}_u \nabla_u, \\
\eta(t, y) &= \rho(t, X(t, y)), \quad v(t, y) \equiv u(t, X(t, y)) \quad \text{and} \quad \Pi(t, y) \equiv p(t, X(t, y)).
\end{align*}
\]

Notice that for any \( t > 0 \), the solution of (1.1) obtained in Theorem 1.1 and Theorem 1.2 satisfies the smoothness assumption of Proposition 2 in [10], so that \( (\eta, v, \Pi) \) defined by (3.10) solves

\[
\begin{cases}
\partial_t \eta = 0, \\
\eta \partial_t v - \Delta_u v + \nabla_u \Pi = 0, \\
\text{div}_u v = 0,
\end{cases}
\]

which is the Lagrangian formulation of (1.1).

Now we transform the regularity information of the solution in the Eulerian coordinates into those in the Lagrangian coordinates.

**Lemma 3.3.** Let \( (\rho, u, \nabla p) \) be the global solution of (1.1) obtained in Theorem 1.2 and \( (\eta, v, \Pi) \) be given by (3.10). Then for any \( t \leq T \) determined by (3.6), one has

\[
\int_0^t \tau^{\frac{3}{2}} \left( \| (\partial_\tau v, \nabla^2 v)(\tau) \|_{L^3}^2 + \| \nabla \Pi(\tau) \|_{L^3}^2 \right) d\tau \leq C,
\]

\[
\| \nabla A \|_{L^\infty(L^3)} + \int_0^t \| \nabla v(\tau) \|_{L^\infty} d\tau \leq Ct^{\frac{1}{2}},
\]

\[
\int_0^t \tau^{\frac{3}{2}} \| \nabla v(\tau) \|_{L^\infty}^2 d\tau \leq C,
\]

for some constant \( C \) depending only on \( c_0, C_0 \) in (1.4) and \( \| u_0 \|_{H^1} \).
Proof. We first deduce from (3.5), (3.6) and (3.1) that

\[ (3.13) \quad \| \nabla_y X(t, \cdot) \|_{L^\infty} \leq \exp \left\{ \int_0^t \| \nabla_x u(\tau) \|_{L^\infty} \, d\tau \right\} \leq e^{\frac{t}{2}}, \]

which together with (3.1) and (3.10) implies that

\[ \int_0^t \| \nabla v(t) \|_{L^\infty} \, dt \leq C t^{\frac{1}{4}} \quad \text{and} \quad \int_0^t \tau^{\frac{3}{2}} \| \nabla v(\tau) \|_{L^\infty}^2 \, d\tau \leq C. \]

Furthermore, thanks to \( \det \left( \frac{\partial X(t,y)}{\partial y} \right) = 1 \), and (1.6), (3.1), one has

\[ \| \nabla^2 v \|_{L_t^2(L^3)} \leq \| \nabla p \|_{L_t^2(L^2)} \| \tau^{\frac{3}{2}} \nabla p \|_{L_t^2(L^6)} \leq C. \]

On the other hand, it follows from the proof of (3.13) that

\[ \| \nabla^2 X(t, \cdot) \|_{L^p} \leq \exp \left\{ C \int_0^t \| \nabla_x u(\tau) \|_{L^\infty} \, d\tau \right\} \int_0^t \| \nabla^2 u(\tau, X(\tau, \cdot)) \|_{L^p} \, d\tau \]

\[ \leq C \int_0^t \| \nabla^2 u(\tau, \cdot) \|_{L^p} \, d\tau \quad \text{for any} \, \ p \in [1, \infty]. \]

In particular, if we take \( p = 3 \) in the above inequality and use (3.1) to get

\[ \| \nabla^2 X(t, \cdot) \|_{L^3} \leq C t^{\frac{1}{4}} \| \nabla^2 u \|_{L_t^2(L^2)} \| \tau^{\frac{3}{2}} \nabla^2 u \|_{L_t^2(L^6)} \leq C t^{\frac{1}{4}}, \]

from which and (3.10), we infer

\[ \| \tau^{\frac{1}{2}} \nabla^2 v \|_{L_t^2(L^3)} \leq C \left( \| \tau^{\frac{1}{2}} \nabla^2 u(\tau, X(\tau, \cdot)) \|_{L_t^2(L^3)} \| \nabla_y X \|_{L_t^\infty(L^\infty)} + \| \tau^{\frac{1}{2}} \nabla_x u \|_{L_t^2(L^\infty)} \| \nabla_y X \|_{L_t^\infty(L^3)} \right) \]

\[ \leq C \left( 1 + \| \nabla^2 u \|_{L_t^2(L^3)} \| \tau^{\frac{3}{2}} \nabla^2 u \|_{L_t^2(L^6)} \right) \leq C. \]

On the other hand, thanks to (3.6), we have for \( t \leq T \)

\[ (3.14) \quad A(t, y) = \nabla_x Y(t, x) = \left( Id + (\nabla_y X(t, y) - Id) \right)^{-1} = \sum_{\ell=0}^{\infty} (-1)^{\ell} \left( \int_0^t \nabla_y u(t', X(t', y)) \, dt' \right)^\ell, \]

for \( x = X(t, y) \), so that

\[ \| \nabla A \|_{L_t^\infty(L^3)} \leq C \| \nabla^2 u(\tau, X(\tau, \cdot)) \|_{L_t^1(L^3)} \| \nabla_y X \|_{L_t^\infty(L^\infty)} \leq C t^{\frac{1}{4}}. \]

Finally, it follows from (1.6), (3.1) and (3.7) that

\[ \| \tau^{\frac{1}{2}} \partial_y v \|_{L_t^2(L^3)} \leq \| \tau^{\frac{1}{2}} \partial_y u \|_{L_t^2(L^3)} + \| \tau^{\frac{3}{2}} u \cdot \nabla u \|_{L_t^2(L^3)} \]

\[ \leq C \left\{ \| \partial_y u \|_{L_t^2(L^2)} \| \tau^{\frac{3}{2}} \nabla \partial_y u \|_{L_t^2(L^2)} \right\} \leq C. \]

This completes the proof of the lemma. \( \square \)
Lemma 3.4. Let \((\rho, u, \nabla p)\) be the global solution of (1.1) obtained in Theorem 1.1 and \((\eta, v, \Pi)\) be given by (3.10). Then for any \(t \leq T\) determined by (3.6) and \(0 \leq \alpha < s\), one has

\[
\int_0^t \tau^{1+\alpha-s} \left( \| (\partial_\tau v, \nabla^2 v)(\tau) \|_{L^2_T}^2 + \| \nabla \Pi(\tau) \|_{L^2_T}^2 \right) d\tau \leq C,
\]

(3.15)

\[
\| \nabla A \|_{L^\infty_T(L^{2+\alpha}\Pi)} \leq C t^{\frac{\alpha-s}{2}},
\]

\[
\int_0^t \| \nabla v(\tau) \|_{L^\infty_T} d\tau \leq C t^{\frac{\alpha-s}{2}},
\]

\[
\int_0^t \tau^{1-\frac{s}{2}} \| \nabla v(\tau) \|_{L^\infty_T}^2 d\tau \leq C,
\]

for some constant \(C\) depending only on \(c_0, C_0\) in (1.2) and \(\| u_0 \|_{H^s}\).

Proof. The proof is similar to Lemma 3.3. We omit the details. \qed

3.3. The proof of the uniqueness. We first recall the following lemma from [10].

Lemma 3.5. Let \(\eta \in L^\infty(\mathbb{R}^d)\) be a time independent positive function, and be bounded away from zero. Let \(R\) satisfy \(R_i \in L^2((0, T) \times \mathbb{R}^d)\) and \(\nabla \div R \in L^2((0, T) \times \mathbb{R}^d)\). Then the following system

\[
\begin{cases}
\eta \partial_t v - \Delta v + \nabla \Pi = f & (t, x) \in (0, T) \times \mathbb{R}^d, \\
\div v = \div R, \\
v|_{t=0} = v_0,
\end{cases}
\]

has a unique solution \((v, \nabla \Pi)\) such that

\[
\| \nabla v \|_{L^\infty_T(L^2)} + \| (v_t, \nabla^2 v, \nabla \Pi) \|_{L^2_T(L^2)} \leq C \left( \| v_0 \|_{L^2} + \| (f, R_t) \|_{L^2_T(L^2)} + \| \nabla \div R \|_{L^2_T(L^2)} \right),
\]

where \(C\) depends on \(\inf \eta\) and \(\sup \eta\), but independent of \(T\).

Proof to the uniqueness parts of Theorems 1.1 and 1.2. Let \((\rho_i, u_i, \nabla p_i), i = 1, 2\), be two solutions of (1.1) obtained in Theorem 1.1 and Theorem 1.2, and \((\eta_i, v_i, \nabla \Pi_i), i = 1, 2\), be determined by (3.10). We denote \(A_i \equiv A(u_i)\) for \(i = 1, 2\), and \(\delta v = v_2 - v_1\), \(\delta \Pi = \Pi_2 - \Pi_1\) and \(\delta A = A_2 - A_1\), then we deduce from (3.11) that

(3.16)

\[
\begin{cases}
\rho_0 \partial_t \delta v - \Delta \delta v + \nabla \delta \Pi = \delta f_1 + \delta f_2, \\
\div \delta v = \div \delta g, \\
\delta v|_{t=0} = 0,
\end{cases}
\]

with

\[
\delta f_1 \equiv - \left[ (\nabla - \nabla u_1) \Pi_1 - (\nabla - \nabla u_2) \Pi_2 \right] = -(I - A_2^t) \delta \Pi - \delta A^t \nabla \Pi_1,
\]

(3.17)

\[
\delta f_2 \equiv - \left[ (\Delta - \Delta u_1) v_1 - (\Delta - \Delta u_2) v_2 \right] = \div \left[ (I - A_2 A_1^t) \delta v + (A_1 A_1^t - A_2 A_2^t) \nabla v_1 \right],
\]

\[
\delta g \equiv -(I - A_1) v_1 + (I - A_2) v_2 = (I - A_2) \delta v - \delta A v_1.
\]

We denote

\[
\delta E(t) \equiv \| \nabla \delta v \|_{L^\infty_T(L^2)} + \| (\partial_t \delta v, \nabla^2 \delta v, \nabla \delta \Pi) \|_{L^2_T(L^2)}.
\]

Then we infer from Lemma 3.5 and (3.16) that

\[
\delta E(t) \leq C \left( \| \delta f_1 \|_{L^2_T(L^2)} + \| \delta f_2 \|_{L^2_T(L^2)} + \| \nabla \div \delta g \|_{L^2_T(L^2)} + \| \partial_t \delta g \|_{L^2_T(L^2)} \right).
\]
We will show that
\[\|\delta f_1\|_{L_t^2(L^2)} + \|\delta f_2\|_{L_t^2(L^2)} + \|\nabla \text{div } \delta g\|_{L_t^2(L^2)} + \|\partial_t \delta g\|_{L_t^2(L^2)} \leq \varepsilon(t)\delta E(t),\]
where the function \(\varepsilon(t)\) tends to zero as \(t\) goes to zero. With (3.18) being granted, we infer that
\[\delta E(t) \leq \varepsilon(t)\delta E(t),\]
which ensures the uniqueness of solutions obtained in Theorem 1.1 and Theorem 1.2 on a sufficiently small time interval \([0, T_1]\). The uniqueness on the whole time \([0, \infty)\) can be obtained by a bootstrap argument. \(\square\)

Now let us turn to the proof (3.18). Indeed thanks to (3.6), we can take the time \(T\) to be small enough so that
\[\int_0^T \|\nabla v_i(\tau)\|_{L^\infty} d\tau \leq \frac{1}{2}, \quad i = 1, 2.\]
As a convention in the sequel, we shall always assume that \(t \leq T\). Thanks to (3.14), we write
\[\delta A(t) = (\int_0^t \nabla \delta v\, d\tau)(\sum_{k \geq 1} \sum_{0 \leq j < k} C_j C_k^{k-j}) \quad \text{with} \quad C_i(t) \equiv \int_0^t \nabla v_i\, d\tau.\]

The proof of (3.18) will split into the following two cases.

**Proof of (3.18) in 3-D case.** We first deduce from (3.14) and (3.12) that
\[\|(Id - A_i^2)\nabla \Pi\|_{L_t^2(L^2)} \leq C \int_0^t \|\nabla g v_2(t')\|_{L^\infty} \|\nabla \Pi\|_{L_t^2(L^2)} \leq C t^{\frac{1}{4}} \delta E(t),\]
While it follows from (3.19) that
\[\|\delta A\|_{L_t^\infty(L^6)} \leq C \int_0^T \|\nabla \delta v\|_{L_t^\infty(L^6)} \leq C t^{\frac{1}{4}} \|\nabla^2 \delta v\|_{L_t^2(L^2)},\]
which along with (3.12) implies
\[\|\delta A^i \nabla \Pi_1\|_{L_t^2(L^2)} \leq \|\tau^{-\frac{1}{4}} \delta A(\tau)\|_{L_t^\infty(L^6)} \|\tau^{\frac{1}{4}} \nabla \Pi_1(\tau)\|_{L_t^2(L^3)} \leq C t^{\frac{1}{4}} \delta E(t),\]
so that thanks to (3.17), we obtain
\[\|\delta f_1\|_{L_t^2(L^2)} \leq C t^{\frac{1}{4}} \delta E(t).\]

Next we handle \(\nabla \text{div } \delta g\). We first get by applying (3.19) that
\[\|\nabla \delta A(t)\| \leq C \int_0^t \|\nabla^2 \delta v\| d\tau + C \int_0^t |\nabla \delta v| d\tau \int_0^t (|\nabla^2 v_1| + |\nabla^2 v_2|) d\tau.\]
Hence, we have by (3.12) that
\[\|\nabla (\delta A \nabla v_1)\|_{L_t^2(L^2)} \leq C \left\{ \int_0^T \|\nabla^2 \delta v\| d\tau \|\nabla v_1\|_{L_t^2(L^2)} + \int_0^T |\nabla \delta v| d\tau \|\nabla^2 v_1\|_{L_t^2(L^2)} \right\} \]
\[\quad + \left\{ \int_0^T \|\nabla^2 \delta v\| d\tau \|\nabla \Pi_1\|_{L_t^2(L^2)} \right\} \]
\[\quad + \left\{ \int_0^T |\nabla \delta v| d\tau \right\} \]
\[\leq C \left\{ \tau^{-\frac{1}{4}} \int_0^T \|\nabla^2 \delta v\|_{L_t^\infty(L^2)} \|\tau^{\frac{1}{4}} \nabla v_1\|_{L_t^2(L^\infty)} \right\} \]
\[\quad + \left\{ \tau^{-\frac{1}{4}} \int_0^T \|\nabla \delta v\|_{L_t^\infty(L^6)} \|\tau^{\frac{1}{4}} (\nabla^2 v_1, \nabla^2 v_2)\|_{L_t^2(L^3)} (1 + \|\tau^{\frac{1}{4}} \nabla v_1\|_{L_t^2(L^\infty)}) \right\} \]
\[\leq C t^{\frac{1}{4}} \delta E(t).\]
Along the same line, one has
\[
\|\nabla((I - A_2)\nabla\delta v)\|_{L^2_t(L^2)}
\leq C\left\{ \int_0^t \|\nabla^2 v_2\|_{L^p_t(L^q)} \|\nabla\delta v\|_{L^q_t(L^p)} + \int_0^t \|\nabla v_2\|_{L^p_t(L^q)} \|\nabla^2 \delta v\|_{L^2_t(L^2)} \right\}
\leq Ct^{\frac{1}{2}}\delta E(t).
\]
Thus, it follows from (3.9) and (3.17) that
\[
\|\nabla\operatorname{div} \delta g\|_{L^2_t(L^2)} \leq Ct^{\frac{1}{2}}\delta E(t).
\]
To deal with $\delta f_2$, we write
\[
(A_2A_2^t - A_1A_1^t)\nabla v_1 = (-\delta A(I - A_2^t) + (I - A_1)\delta A^t)\nabla v_1 + (\delta A^t + \delta A)\nabla v_1.
\]
It is easy to check that
\[
\|\operatorname{div}\left[(\delta A(I - A_2^t))\nabla v_1\right]\|_{L^2_t(L^2)}
\leq \|\tau^{-\frac{1}{2}}\nabla \delta A\|_{L^\infty_t(L^2)} \|I - A_2\|_{L^\infty_t(L^\infty)} \|\tau^{\frac{1}{2}}\nabla v_1\|_{L^2_t(L^2)}
\quad + \|\tau^{-\frac{1}{2}}\delta A\|_{L^\infty_t(L^3)} \|\nabla A_2\|_{L^\infty_t(L^3)} \|\tau^{\frac{1}{2}}\nabla v_1\|_{L^2_t(L^2)}
\quad + \|\tau^{-\frac{1}{2}}\delta A\|_{L^\infty_t(L^3)} \|I - A_2\|_{L^\infty_t(L^\infty)} \|\tau^{\frac{1}{2}}\nabla^2 v_1\|_{L^2_t(L^2)},
\]
which along with (3.21) and (3.12) implies that
\[
\|\operatorname{div}\left[(\delta A(I - A_2^t))\nabla v_1\right]\|_{L^2_t(L^2)}
\leq Ct^{\frac{1}{2}}\|\nabla^2 \delta v\|_{L^2_t(L^2)} \left\{ \|\nabla v_2\|_{L^x_t(L^q)} \left( \|\tau^{\frac{1}{2}}\nabla v_1\|_{L^2_t(L^2)} + \|\tau^{\frac{1}{2}}\nabla^2 v_1\|_{L^2_t(L^2)} \right) \right\}
\quad + \|\nabla A_2\|_{L^\infty_t(L^3)} \|\tau^{\frac{1}{2}}\nabla v_1\|_{L^2_t(L^2)} \leq Ct^{\frac{1}{2}}\delta E(t).
\]
The same estimate holds for $\operatorname{div}\left[(I - A_1)\delta A^t)\nabla v_1\right]$ and $\operatorname{div}\left[(\delta A^t + \delta A)\nabla v_1\right]$. Hence,
\[
\|\operatorname{div}\left[(A_2A_2^t - A_1A_1^t)\nabla v_1\right]\|_{L^2_t(L^2)} \leq Ct^{\frac{1}{2}}\delta E(t).
\]
To handle $\operatorname{div}\left[(I - A_2A_2^t)\nabla \delta v\right]$, we write
\[
(I - A_2A_2^t)\nabla \delta v = -\{(I - A_2)(I - A_2^t) - (I - A_2^t) - (I - A_2)\}\nabla \delta v.
\]
Then we get, by using (3.12), that
\[
\|\operatorname{div}\left[(I - A_2A_2^t)\nabla \delta v\right]\|_{L^2_t(L^2)}
\leq C\left\{ \|\nabla A_2\|_{L^\infty_t(L^3)} \|I - A_2\|_{L^\infty_t(L^\infty)} \|\nabla \delta v\|_{L^2_t(L^2)} + \|\nabla \delta v\|_{L^2_t(L^2)} \|\nabla^2 \delta v\|_{L^2_t(L^2)} \right\}
\leq Ct^{\frac{1}{2}}\delta E(t).
\]
Summing up, we obtain
\[
\|\delta f_2\|_{L^2_t(L^2)} \leq Ct^{\frac{1}{2}}\delta E(t).
\]
Finally, let us turn to the estimate of $\|\partial_t \delta g\|_{L_t^2(L^2)}$. Thanks to (3.19), we have
\[
\|\partial_t [\delta A v_1]\|_{L_t^2(L^2)} \leq C \left\{ \|\nabla \delta v\|_{L_t^2(L^2)} \|v_1\|_{L_t^2(L^2)} + \left| \int_0^T |\nabla \delta v\| \partial t v_1\|_{L_t^2(L^2)} \right| \right\} \\
\leq C \left\{ \|\nabla \delta v\|_{L_t^2(L^2)} \|v_1\|_{L_t^2(L^2)} + \right| \|\nabla \delta v\| \partial t v_1\|_{L_t^2(L^2)} \right| \right\}
\]
which together with (1.6) and (3.12) ensures that
\[
\|\partial_t [\delta A v_1]\|_{L_t^2(L^2)} \leq C \left\{ \|\nabla u_1\|_{L_t^2(L^2)} \|\Delta u_1\|_{L_t^2(L^2)} \|\nabla \delta v\|_{L_t^2(L^2)} + t^{\frac{1}{2}} \|\nabla ^2 \delta v\|_{L_t^2(L^2)} \\
\times \left( \|\nabla \delta v\|_{L_t^2(L^2)} \|v_1\|_{L_t^2(L^2)} \|\nabla \delta v\|_{L_t^2(L^2)} + \|\nabla \delta v\|_{L_t^2(L^2)} \right) \right\}
\]
\[
\leq Ct^{\frac{1}{2}} \|\nabla \delta v\|_{L_t^2(L^2)} + Ct^{\frac{1}{2}} \|\nabla ^2 \delta v\|_{L_t^2(L^2)} \leq Ct^{\frac{1}{2}} \delta E(t).
\]
Along the same line, one has
\[
\|\partial_t [(\text{Id} - A_2) \delta v]\|_{L_t^2(L^2)} \leq C \left\{ \|\nabla v_2 \delta v\|_{L_t^2(L^2)} + \left| \int_0^T |\nabla v_2\| \partial t v_2\|_{L_t^2(L^2)} \right| \right\} \\
\leq C \left\{ \|\nabla \delta v\|_{L_t^2(L^2)} \|v_2\|_{L_t^2(L^2)} + \|\nabla \delta v\|_{L_t^2(L^2)} \right| \right\}
\]
\[
\leq Ct^{\frac{1}{2}} \delta E(t).
\]
Hence, we arrive at
\[
(3.26) \quad \|\partial_t \delta g\|_{L_t^2(L^2)} \leq Ct^{\frac{1}{2}} \delta E(t).
\]
Then (3.18) follows from (3.20), (3.22), (3.25) and (3.26).

**Proof of (3.18) in 2-D case.** In what follows, we shall always take $0 < \alpha < s$. By virtue of (3.14) and (3.15), we have
\[
\|[(\text{Id} - A_2^1) \nabla \Pi\|_{L_t^2(L^2)} \leq C \int_0^t \|\nabla v_2(t')\|_{L^2} \|\nabla \Pi\|_{L_t^2(L^2)} \leq Ct^{\frac{1}{2}} \delta E(t),
\]
and by Gagliardo-Nirenberg inequality, one has
\[
\|\delta A\|_{L_t^2(\mathbb{H}^+) \leq C \int_0^t \|\nabla \delta v\|_{L_t^2(\mathbb{H}^+)} \leq C \int_0^t \|\nabla \delta v\|_{L_t^2(\mathbb{H}^+)} \|\nabla \delta v\|_{L_t^2(\mathbb{H}^+)} \|\nabla \delta v\|_{L_t^2(\mathbb{H}^+)} \|\partial_t \delta v\|_{L_t^2(L^2)} \right| \right\}
\]
\[
\leq C \left\{ \|\nabla \delta v\|_{L_t^2(\mathbb{H}^+)} \|\nabla \delta v\|_{L_t^2(\mathbb{H}^+)} \|\nabla \delta v\|_{L_t^2(\mathbb{H}^+)} \right| \right\}
\]
\[
\leq Ct^{\frac{1}{2}} \delta E(t),
\]
\[
\|\delta A^1 \nabla \Pi\|_{L_t^2(L^2)} \leq \|\nabla \Pi\|_{L_t^2(L^2)} \|\nabla \Pi\|_{L_t^2(L^2)} \leq Ct^{\frac{1}{2}} \delta E(t).
\]
As a consequence, we obtain
\[
(3.28) \quad \|\delta f_1\|_{L_t^2(L^2)} \leq Ct^{\frac{1}{2}} \delta E(t).
\]
Whereas thanks to (3.24), we get, by using (3.15), that

\[
\|\nabla(\delta A \nabla v_1)\|_{L_t^2(L^2)} \leq C \left\lbrace \int_0^T \|\nabla^2 \delta v \|_{L_t^2(L^2)} \, dt + \int_0^T \|\nabla \delta v \|_{L_t^2(L^2)} \, dt \right\rbrace
\]

and we also have

\[
\|\nabla((\text{Id} - A_2) \nabla \delta v)\|_{L_t^2(L^2)} 
\leq C \left\lbrace \int_0^T \|\nabla^2 \delta v \|_{L_t^2(L^2)} \, dt + \int_0^T \|\nabla \delta v \|_{L_t^2(L^2)} \, dt \right\rbrace
\]

Here we used the fact that

\[
\|\nabla \delta v\|_{L_t^2(L^\infty)} \leq C t^{\frac{\alpha}{2(1+\alpha)}} \|\nabla^2 \delta v\|_{L_t^2(L^2)} \|\nabla v\|_{L_t^2(L^\infty)}^{1-\alpha} \leq C t^{\frac{\alpha}{2(1+\alpha)}} \delta E(t).
\]

Hence, we obtain

\[
\|\nabla \text{div} \delta g\|_{L_t^2(L^2)} \leq C t^{\frac{\alpha}{2(1+\alpha)}} \delta E(t).
\]

Now we handle \(\|\delta f_2\|_{L_t^2(L^2)}\). Indeed it follows from (3.15) and (3.27) that

\[
\|\text{div} \left[ (\delta A(\text{Id} - A_2') \nabla v_1) \right]\|_{L_t^2(L^2)} 
\leq C \left\lbrace \int_0^T \|\nabla^2 \delta v \|_{L_t^2(L^2)} \, dt + \int_0^T \|\nabla \delta v \|_{L_t^2(L^2)} \, dt \right\rbrace
\]

The same estimate holds for the remaining terms in (3.23). Hence, we get

\[
\|\text{div} \left[ (A_2 A_2 - A_1 A_1') \nabla v_1 \right]\|_{L_t^2(L^2)} \leq C t^{\frac{\alpha}{2(1+\alpha)}} \delta E(t).
\]

Whereas thanks to (3.24), we get, by using (3.15), that

\[
\|\text{div}((\text{Id} - A_2 A_2') \nabla \delta v)\|_{L_t^2(L^2)} 
\leq C \left( \|\nabla A_2\|_{L_t^2(L^2)} \|\text{Id} - A_2\|_{L_t^2(L^\infty)} \|\nabla \delta v\|_{L_t^2(L^\infty)} + \|\nabla \delta v\|_{L_t^2(L^\infty)} \|\nabla^2 \delta v\|_{L_t^2(L^2)} \right)
\]

So we obtain

\[
\|\delta f_2\|_{L_t^2(L^2)} \leq C t^{\frac{\alpha}{2(1+\alpha)}} \delta E(t).
\]
Finally we deal with $\|\partial_t \delta v\|_{L^2_t(L^2)}$. As in the 3-D case, we have

$$
\|\partial_t [\delta Av_1]\|_{L^2_t(L^2)} \leq C\left\{ \|\nabla \delta v\|_{L^2_t(L^2)}^2 + \int_0^T |\nabla \delta v| \, dt \|\nabla v_1, \nabla v_2\|_{L^2_t(L^2)} \right\}
$$

By (3.15), the first term on the right hand side is bounded by

$$
\|\nabla \delta v\|_{L^\infty_t(L^2)} \|v_1\|_{L^2_t(L^2)} \leq C\delta E(t)\|v_1\|_{L^2_t(L^2)}^2 \leq C\tilde{t}\delta E(t),
$$

and the second term is bounded by

$$
\left\| \tau^{-\frac{1+\alpha-s}{2}} \int_0^\tau |\nabla \delta v| \, dt \right\|_{L^\infty_t(L^\frac{2}{2})} \tau^{\frac{1+\alpha-s}{2}} \|\partial_t v_1\|_{L^2_t(L^\frac{2}{2})} \leq C\tilde{t}\delta E(t),
$$

and the second term is bounded by

$$
\left\| \tau^{-\frac{1+\alpha-s}{2}} \int_0^\tau |\nabla \delta v| \, dt \right\|_{L^\infty_t(L^\frac{2}{2})} \tau^{-\frac{1+\alpha-s}{2}} \|\nabla \delta v\|_{L^2_t(L^\frac{2}{2})} \leq C\tilde{t}\delta E(t),
$$

On the other hand, it follows from Gagliardo-Nirenberg inequality that

$$
\left\| \tau^{-\frac{1-s}{2}} \delta v\right\|_{L^\infty_t(L^\frac{2}{2})} \left\leq C\|\tau^{-\frac{1-s}{2}} \delta v\|_{L^\infty_t(L^2)}^{\frac{1}{\alpha}} \|\nabla \delta v\|_{L^2_t(L^2)}^{1-\alpha} \leq C\tilde{t}\delta E(t),
$$

$$
\left\| \tau^{-\frac{1-s}{2}} \nabla v_2\delta v\|_{L^2_t(L^\frac{2}{2})} \leq C\|\nabla v_2\|_{L^2_t(L^2)}^{1-\alpha} \|\tau^{-\frac{1-s}{2}} \nabla v_2\|_{L^2_t(L^2)}^{\alpha} \leq C,
$$

from which and (3.15), we infer that

$$
\|\partial_t [(Id - A_2)\delta v]\|_{L^2_t(L^2)} \leq C\left\{ \left\| \tau^{-\frac{1-s}{2}} \delta v\right\|_{L^\infty_t(L^2)} \left\| \tau^{\frac{1-s}{2}} \nabla v_2\right\|_{L^2_t(L^\frac{2}{2})} \right\} + \left\| \tau^{\frac{1-s}{2}} \delta v\|_{L^\infty_t(L^\frac{2}{2})} \left\| \tau^{-\frac{1-s}{2}} \nabla v_2\|_{L^2_t(L^\frac{2}{2})} + \|\partial_t \delta v\|_{L^2_t(L^2)} \right\}
$$

$$
\leq C\tilde{t}\delta E(t).
$$

Therefore, we arrive at

$$
(3.32) \|\delta g\|_{L^2_t(L^2)} \leq C\tilde{t}\delta E(t).
$$

Then (3.18) in the 2-D case follows from (3.28)-(3.32).

\[\square\]

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References


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