ON THE FREE BOUNDARY PROBLEM OF THREE-DIMENSIONAL COMPRESSIBLE EULER EQUATIONS IN PHYSICAL VACUUM

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ABSTRACT. In this paper, we establish a priori estimates for the three-dimensional compressible Euler equations with moving physical vacuum boundary, the $\gamma$-gas law equation of state for $\gamma = 2$ and the general initial density $\rho_0 \in H^6(\Omega)$. Because of the degeneracy of the initial density, we investigate the estimates of the horizontal spatial and time derivatives and then obtain the estimates of the normal or full derivatives through the elliptic-type estimates. We derive a mixed space-time interpolation inequality which play a vital role in our energy estimates. The results improve the known results (Comm. Math. Phys. 296 (2010) 559-587) on a priori estimates for the case $\gamma = 2$.

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1. INTRODUCTION

In the present paper, we consider the following compressible Euler equations

\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0 \quad \text{in } \Omega(t) \times (0, T], \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla p &= 0 \quad \text{in } \Omega(t) \times (0, T], \\
p &= 0 \quad \text{on } \Gamma_1(t) \times (0, T], \\
u_3 &= 0 \quad \text{on } \Gamma_0 \times (0, T], \\
\partial_t \Gamma_1(t) = \mathcal{V} (\Gamma_1(t)) &= u \cdot \mathcal{N} \quad \text{in } (0, T], \\
(\rho, u) &= (\rho_0, u_0) \quad \text{in } \Omega \times \{t = 0\},
\end{align*}

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\[\Omega(0) = \Omega, \Gamma_1(0) = \Gamma_1, \quad (1.1g)\]

where \(\rho\) denotes the density, the vector field \(u = (u_1, u_2, u_3) \in \mathbb{R}^3\) denotes the Eulerian velocity field, and \(p\) denotes the pressure function. \(\nabla = (\partial_1, \partial_2, \partial_3)\) and \(\text{div}\) are the usual gradient operator and spatial divergence where \(\partial_i = \partial / \partial x_i\). The open, bounded domain \(\Omega(t) \subset \mathbb{R}^3\) denotes the changing volume occupied by the fluid, \(\Gamma_1(t) := \partial \Omega(t)\) denotes the moving vacuum boundary, \(\mathcal{V}(\Gamma_1(t))\) denotes the normal velocity of \(\Gamma_1(t)\), \(\mathcal{N}\) denotes the outward unit normal vector to the boundary \(\Gamma_1(t)\), and \(\Gamma_0\) is a fixed boundary. The equation of the pressure \(p(\rho)\) is given by

\[p(x, t) = C_\gamma \rho^\gamma(x, t), \quad (1.2)\]

where \(\gamma\) is the adiabatic index and we will only consider the case \(\gamma = 2\) in this paper; \(C_\gamma\) is the adiabatic constant which we set to unity, i.e., \(C_\gamma = 1\); and

\[\rho > 0 \quad \text{in } \Omega(t) \quad \text{and} \quad \rho = 0 \quad \text{on } \Gamma_1(t). \quad (1.3)\]

Equation (1.1a) is the conservation of mass, (1.1b) is the conservation of momentum. The boundary condition (1.1c) states that pressure (and hence density) vanishes along the vacuum boundary, (1.1d) describes that the normal component of the velocity vanishes on the fixed boundary \(\Gamma_0\), (1.1e) indicates that the vacuum boundary is moving with the normal component of the fluid velocity, (1.1f) and (1.1g) are the initial conditions for the density, velocity, domain and boundary.

To avoid the use of local coordinate charts necessary for arbitrary geometries, for simplicity, we assume that the initial domain \(\Omega \subset \mathbb{R}^3\) at time \(t = 0\) is given by

\[\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \mathbb{T}^2, x_3 \in (0, 1)\},\]

where \(\mathbb{T}^2\) denotes the 2-torus, which can be thought of as the unit square with periodic boundary conditions. This permits the use of one global Cartesian coordinate system. At \(t = 0\), the reference vacuum boundary is the top boundary

\[\Gamma_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \mathbb{T}^2, x_3 = 1\},\]

while the bottom boundary

\[\Gamma_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \mathbb{T}^2, x_3 = 0\}\]

is fixed with the boundary condition (1.1d).

We set the unit normal vectors \(N = (0, 0, 1)\) on \(\Gamma_1\) and \(N = (0, 0, -1)\) on \(\Gamma_0\). We use the standard basis on \(\mathbb{R}^3\): \(e_1 = (1, 0, 0), e_2 = (0, 1, 0)\) and \(e_3 = (0, 0, 1)\). Similarly, the unit tangent vectors on \(\Gamma = \Gamma_0 \cup \Gamma_1\) are given by

\[T_1 = (1, 0, 0) \quad \text{and} \quad T_2 = (0, 1, 0).\]

Throughout the paper, we will use Einstein’s summation convention. The \(k^{\text{th}}\)-partial derivative of \(F\) will be denoted by \(F_k = \frac{\partial F}{\partial x^k}\). Then we have

\[\text{div}(\rho u \otimes u)^i = (\rho u^i u^j) = \text{div}(\rho u^i u^j) + \rho u \cdot \nabla u^i,\]

which yields that (1.1b) can be rewritten, in view of (1.1a), as

\[\rho (\partial_t u + u \cdot \nabla u) + \nabla p = 0.\]

Thus, the system (1.1) can be rewritten as

\[
\begin{align*}
\text{div}(\rho u) &= 0 & \text{in } \Omega(t) \times (0, T], & \quad (1.4a) \\
\rho (\partial_t u + u \cdot \nabla u) + \nabla p &= 0 & \text{in } \Omega(t) \times (0, T], & \quad (1.4b) \\
p &= 0 & \text{on } \Gamma_1(t) \times (0, T], & \quad (1.4c) \\
u_3 &= 0 & \text{on } \Gamma_0 \times (0, T], & \quad (1.4d)
\end{align*}
\]
\[
\begin{align*}
\partial_t \Gamma_1(t) &= u \cdot \mathcal{N} & \text{in } (0, T], & \quad (1.4e) \\
(\rho, u) &= (\rho_0, u_0) & \text{in } \Omega \times \{t = 0\}, & \quad (1.4f) \\
\Omega(0) &= \Omega, \quad \Gamma_1(0) = \Gamma_1. & \quad (1.4g)
\end{align*}
\]

With the sound speed given by \( c := \sqrt{\frac{\partial p}{\partial \rho}} \) and \( N \) denoting the outward unit normal to \( \Gamma_1 \), the satisfaction of the condition
\[
-\infty < \frac{\partial c^2}{\partial N} < 0
\]
in a small neighborhood of the boundary defines a physical vacuum boundary (cf. [12]), where \( c_0 = c|_{t=0} \) denotes the initial sound speed of the gas. In other words, the pressure accelerates the boundary in the normal direction. The physical vacuum condition (1.5) for \( \gamma = 2 \) is equivalent to the requirement
\[
\frac{\partial \rho_0}{\partial N} < 0 \quad \text{on } \Gamma_1. \tag{1.6}
\]

Since \( \rho_0 > 0 \) in \( \Omega \), (1.6) implies that for some positive constant \( C \) and \( x \in \Omega \) near the vacuum boundary \( \Gamma_1 \),
\[
\rho_0(x) \geq C \text{dist}(x, \Gamma_1), \tag{1.7}
\]
where \( \text{dist}(x, \Gamma_1) \) denotes the distance of \( x \) away from \( \Gamma_1 \).

The moving boundary is characteristic because of the evolution law (1.1e), and the system of conservation laws is degenerate because of the appearance of the density function as a coefficient in the nonlinear wave equation which governs the dynamics of the divergence of the velocity of the gas. In turn, weighted estimates show that this wave equation indeed loses derivatives with respect to the uniformly hyperbolic non-degenerate case of a compressible liquid, wherein the density takes the value of a strictly positive constant on the moving boundary [2]. The condition (1.7) violates the uniform Kreiss-Lopatinskii condition [9] because of resonant wave speeds at the vacuum boundary for the linearized problem. The methods developed for symmetric hyperbolic conservation laws would be extremely difficult to implement for this problem, wherein the degeneracy of the vacuum creates further difficulties for the linearized estimates.

Now, we transform the system (1.4) in terms of Lagrangian variables. Let \( \eta(x, t) \) denote the “position” of the gas particle \( x \) at time \( t \). Thus,
\[
\partial_t \eta = u \circ \eta \quad \text{for } t > 0, \quad \text{and} \quad \eta(x, 0) = x, \tag{1.8}
\]
where \( \circ \) denotes the composition, i.e., \( (u \circ \eta)(x, t) := u(\eta(x, t), t) \).

We let
\[
v = u \circ \eta, \quad f = \rho \circ \eta, \quad A = (\nabla \eta)^{-1}, \quad J = \det \nabla \eta, \quad a = JA,
\]
where \( A \) is the inverse of the deformation tensor
\[
\nabla \eta = (\eta_{i,j}) = \begin{pmatrix}
\eta_{1,1} & \eta_{1,2} & \eta_{1,3} \\
\eta_{2,1} & \eta_{2,2} & \eta_{2,3} \\
\eta_{3,1} & \eta_{3,2} & \eta_{3,3}
\end{pmatrix},
\]
\[
J \text{ is the Jacobian determinant and } a \text{ is the classical adjoint of } \nabla \eta, \text{ i.e., the transpose of the cofactor matrix of } \nabla \eta, \text{ explicitly,}
\]
\[
a = \begin{pmatrix}
\eta_{2,2} \eta_{3,3} - \eta_{2,3} \eta_{3,2} & \eta_{1,3} \eta_{3,2} - \eta_{1,2} \eta_{3,3} & \eta_{1,2} \eta_{2,3} - \eta_{1,3} \eta_{2,2} \\
\eta_{2,3} \eta_{3,1} - \eta_{2,1} \eta_{3,3} & \eta_{1,3} \eta_{3,1} - \eta_{1,1} \eta_{3,3} & \eta_{1,1} \eta_{2,3} - \eta_{1,3} \eta_{2,1} \\
\eta_{2,1} \eta_{3,2} - \eta_{2,2} \eta_{3,1} & \eta_{1,2} \eta_{3,1} - \eta_{1,1} \eta_{3,2} & \eta_{1,1} \eta_{2,2} - \eta_{1,2} \eta_{2,1}
\end{pmatrix}. \tag{1.9}
\]
Since on the fixed boundary $\Gamma_0$, $\eta^3 = x_3 = 0$ so that according to (1.9), the components $a_1^3 = a_2^3 = 0$ on $\Gamma_0$, and $v_3 = 0$ on $\Gamma_0$ due to $v \cdot (0,0,-1) = 0$ where $(0,0,-1)$ is the outward unit normal vector to $\Gamma_0$, then the Lagrangian version of (1.4) can be written in the fixed reference domain $\Omega$ as

$$
\begin{align*}
  f_t + fA_i^jv_{,j} &= 0 & \text{in } \Omega \times (0, T], & (1.10a) \\
  f v_i^j + A_i^j f^2_{,j} &= 0 & \text{in } \Omega \times (0, T], & (1.10b) \\
  f &= 0 & \text{on } \Gamma_1 \times (0, T], & (1.10c) \\
  a_1^3 = a_2^3 &= 0, \quad v_3 = 0 & \text{on } \Gamma_0 \times (0, T], & (1.10d) \\
  (\rho, v, \eta) &= (\rho_0, u_0, e) & \text{in } \Omega \times \{ t = 0 \}, & (1.10e)
\end{align*}
$$

where $e(x) = x$ denotes the identity map on $\Omega$.

From the derivative formula of determinants, we have

$$
J_t = JA_i^jv_{,j} = a_i^j v_{,j}.
$$

(1.11)

It follows from (1.10a) and (1.11) that

$$
f_t + fJ^{-1}J_t = 0, \quad \text{or } \partial_t (fJ) = 0, \quad \text{i.e., } f = \rho_0J^{-1},
$$

(1.12)

thus, the initial density function $\rho_0$ can be viewed as a parameter in compressible Euler equations.

Using the identity $A_i^j = J^{-1}a_i^j$, we write (1.10) as

$$
\begin{align*}
  \rho_0 v_i^j + a_i^j (\rho_0^2J^{-2})_{,j} &= 0 & \text{in } \Omega \times (0, T], & (1.13a) \\
  \rho_0 &= 0 & \text{on } \Gamma_1, & (1.13b) \\
  a_1^3 = a_2^3 &= 0, \quad v_3 = 0 & \text{on } \Gamma_0 \times (0, T], & (1.13c) \\
  (v, \eta) &= (u_0, e) & \text{in } \Omega \times \{ t = 0 \}, & (1.13d)
\end{align*}
$$

with $\rho_0(x) \geq C\text{dist}(x, \Gamma_1)$ for $x \in \Omega$ near $\Gamma_1$.

To understand the behavior of vacuum states is an important problem in gas and fluid dynamics. In particular, the physical vacuum, in which the boundary moves with a nontrivial finite normal acceleration, naturally arises in the study of the motion of gaseous stars or shallow water [8]. Despite its importance, there are only few mathematical results available near vacuum. The main difficulty lies in the fact that the physical systems become degenerate along the vacuum boundary. The existence and uniqueness for the three-dimensional compressible Euler equations modeling a liquid rather than a gas was established in [11] where the density is positive on the vacuum boundary. Trakhinin provided an alternative proof for the existence of a compressible liquid, employing a solution strategy based on symmetric hyperbolic systems combined with the Nash-Moser iteration in [14].

The local existence for the physical vacuum singularity can be found in the recent papers by Jang and Masmoudi [7, 8] and by Coutand and Shkoller [5, 6] for the one-dimensional and three-dimensional compressible gases. Coutand, Lindblad and Shkoller [3] established a priori estimates based on time differentiated energy estimates and elliptic estimates for normal derivatives for $\gamma = 2$ with $\rho_0 \in H^4(\Omega)$ where the energy function was given by

$$
\begin{align*}
  E(t) := 4 \sum_{t=0}^{4} \| \partial_t^2 \eta(t) \|_{L^2}^2 + 4 \sum_{t=0}^{4} \left[ \| \rho_0 \partial_t^{4-\ell} \partial_t^2 \nabla \eta(t) \|_0^2 + \| \sqrt{\rho_0} \partial_t^{4-\ell} \partial_t^2 v(t) \|_0^2 \right] \\
  + 3 \| \rho_0 \partial_t^2 f(t) \|_{L^2}^2 + \| \text{curl} \eta \cdot v(t) \|_3^2 + \| \rho_0 \partial_t \text{curl} \eta \cdot v(t) \|_0^2.
\end{align*}
$$
We will not attempt to address exhaustive references in this paper. For more related references, we refer the interested reader to [6, 8] and references therein for a nice history of the analysis of compressible Euler equations.

In the present paper, we will improve the results given in [3, 6] by using a similar argument therein and fully exploiting the degeneracy of the initial density $\rho_0$ and the physical vacuum condition (1.7), especially focusing mainly on the following improvement comparing with [3]. Firstly, the a priori estimates were given in [3] based on a proof for a special case $\rho_0 = 1 - x_3$ of the density, instead of general densities, from which one can not easily obtain the exact Sobolev space that the density should belong to from this special density. Because it will appear a term $\overline{\partial} r \rho_0 v_i^j$ evidently for general $\rho_0$ in the $r$-th order horizontal energy estimates, we must assume $\rho_0 \in H^{r+1}(\Omega)$ at least in view of the higher order Hardy inequality given in Section 2.2, for instance, $\rho_0 \in H^3(\Omega)$ if $r = 4$, see more details in Step 4 of the proof of Proposition 6.1. Secondly, we will prove, in details, a mixed space-time interpolation inequality under the framework of Lebesgue spaces for time but Sobolev spaces for spatial variables in bounded domain used therein and fully exploiting the degeneracy of the initial density $\rho_0$. Hence, (1.7), especially focusing mainly on the following improvement comparing with [3].

Now, we state our main result as follows.

**Theorem 1.1.** Let $\gamma = 2$. Suppose that $(\eta(t), v(t))$ is a smooth solution of the system (1.8) and (1.13) on a time interval $[0, T']$ satisfying the initial bounds $E(0) < \infty$ and

$$V := \sum_{m=0}^{4} \| \partial_t^{2m} D^{4-m} v(0) \|_{L^2(\Omega)} < \infty,$$

and that the initial density $\rho_0 > 0$ in $\Omega$ and $\rho_0 \in H^6(\Omega)$ satisfies the physical vacuum condition (1.7). Then there exists a $T > 0$ so small that the energy function $E(t)$ constructed from the...
solution \((\eta(t), v(t))\) satisfies the a priori estimate
\[
\sup_{[0, T]} E(t) \leq M_0,
\]
where both \(M_0\) and \(T\) are functions of \(E(0)\) and \(V\).

Remark 1.2. The same arguments and the results hold true if the bottom boundary \(\Gamma_0\) is also a moving vacuum boundary, i.e., by changing the boundary condition (1.1d) into \(p = 0\) on \(\Gamma_0(t) \times (0, T]\), which will not cause any additional difficulties except for the transformation of coordinates.

Remark 1.3. For the general cases \(\gamma > 1\) with general densities, we give some further remarks. We think that they are much more different from the special case \(\gamma = 2\). They need to reform the energy function in order to get a priori estimates. For the cases \(\gamma > 2\), it seems to be similar to the case \(\gamma = 2\) due to \(\gamma - 1 > \gamma/2\) in view of the exponent of the weight \(\rho_0^{\gamma/2}\) and \(\rho_0^{\gamma-1}\) and weighted Sobolev embedding relations given in Section 2.1, but it is not easy to deal with the weight \(\rho_0^{\gamma/2}\) in energy estimates in view of the higher order Hardy inequality. For the cases \(1 < \gamma < 2\), one have to use the weight \(\rho_0^{\gamma-1}\) instead of \(\rho_0^{\gamma/2}\) in constructing the energy function according to the physical vacuum condition, the higher order Hardy inequality and weighted Sobolev embedding relations, especially for the cases \(3/2 < \gamma < 2\), however one must deal with many extra, important and difficult remainder integrals in the estimates of every horizontal, time or mixed derivatives. For the cases \(1 < \gamma < 3/2\), it might be different from and difficult than the above cases. We will discuss the above general cases in forthcoming papers if possible.

The rest of this paper is organized as follows. We will give some preliminaries in Section 2. Precisely, we introduce some notations and weighted Sobolev spaces in Section 2.1; we recall the higher-order Hardy-type inequality and Hodge’s decomposition elliptic estimates in Section 2.2; we give the properties of the determinant \(J\), the inverse of the deformation tensor \(A\) and the transpose of the cofactor matrix \(a\) in Section 2.3. Then, we give a mixed space-time interpolation inequality in Section 3, and derive the zero-th order energy estimates in Section 4 and the curl estimates in Section 5. Since the standard energy method is very problematic due to the degeneracy of \(\rho_0\), we first derive the estimates of the horizontal and time derivatives in Sections 6-8 and then obtain the estimates of normal or full derivatives through the elliptic-type estimates in Section 9. Finally, we will complete the proof of the a priori estimates in Section 10.

2. Preliminaries

2.1. Notations and weighted Sobolev spaces. Throughout the paper, we will use the following notation: two-dimensional gradient vector or horizontal derivative \(\mathbf{\partial} = (\partial_1, \partial_2)\), the \(H^s(\Omega)\) interior norm \(\|\cdot\|_s\), and \(H^s(\Gamma)\) boundary norm \(|\cdot|_s\). The component of a matrix \(M\) at the \(i\)th row and the \(j\)th column will be denoted by \(M_{ij}\). Sometimes, we will use “\(\lesssim\)" to stand for “\(\leq C\)" with a generic constant \(C\).

We make use of the Levi-Civita permutation symbol
\[
\varepsilon_{ijk} = \begin{cases} 
1 & \text{even permutation of } \{1,2,3\}, \\
-1 & \text{odd permutation of } \{1,2,3\}, \\
0 & \text{otherwise,}
\end{cases}
\]
and the basic identity regarding the \(i\)th component of the curl of a vector field \(u\):
\[
(curl u)_i = \varepsilon_{ijk} u^k_{,j},
\]
where it means that we have taken the sum with respect to the repeated scripts \( j \) and \( k \). Since \( v_{i,j} = \frac{\partial u_i}{\partial \eta^j} \cdot \frac{\partial \eta^k}{\partial x^j} \), it follows that \( \frac{\partial u_i}{\partial \eta^j} = \frac{\partial x^i}{\partial \eta^j} \cdot v_{i,j} \), i.e., \( u_{i,k} = A_{kj}^i v_{i,j} \). The chain rule shows that

\[
(curl(u)(\eta))_i = (curl_{\eta} v)_i := \epsilon_{ijk} v^k A^i_j,
\]

where the right-hand side defines the Lagrangian curl operator \( curl_{\eta} \). Similarly, we have

\[
div u(\eta) = div_{\eta} v := v^i A^i_j,
\]

and the right-hand side defines the Lagrangian divergence operator \( div_{\eta} \). We also use the notation for any vector field \( F \)

\[
(\nabla_{\eta} F)_j = F^i_j A^i_j.
\]

For any vector field \( F \), we have

\[
curl_{\eta} F = curl(FA) = \left( \begin{array}{c} F^3_j A^i_2 - F^2_j A^i_3 \\ F^1_j A^i_3 - F^3_j A^i_1 \\ F^2_j A^i_1 - F^1_j A^i_2 \end{array} \right),
\]

and then

\[
|\nabla_{\eta} F|^2 = (F^3_j A^i_2 - F^2_j A^i_3)^2 + (F^1_j A^i_3 - F^3_j A^i_1)^2 + (F^2_j A^i_1 - F^1_j A^i_2)^2
\]

\[
= F^3_j A^i_2 F^3_A^i_2 - F^2_j A^i_3 F^2_A^i_3 + F^1_j A^i_3 F^1_A^i_1 - F^3_j A^i_1 F^3_A^i_1 - F^2_j A^i_2 F^2_A^i_2 + F^1_j A^i_2 F^1_A^i_1
\]

\[
- 2 F^3_j A^i_2 F^2_A^i_2 - 2 F^1_j A^i_3 F^2_A^i_2 + 2 F^2_j A^i_1 F^3_A^i_1 - 2 F^1_j A^i_2 F^3_A^i_1 - 2 F^2_j A^i_1 F^1_A^i_1
\]

\[
= F^i_j A^i_n \cdot F^i_n A^i_k - F^i_j A^i_n \cdot F^i_n A^i_l
\]

\[
= F^i_j A^i_n (F^i_n A^i_k - F^i_n A^i_l) = |\nabla_{\eta} F|^2 - F^i_j A^i_n \cdot F^i_n A^i_l,
\]

namely,

\[
|\nabla_{\eta} F|^2 = |curl_{\eta} F|^2 + (\nabla_{\eta} F) \cdot (\nabla_{\eta} F)^T,
\]

where the superscript \( T \) denotes the transpose of the matrix.

As a generalization of the standard nonlinear Gronwall inequality, we introduce a polynomial-type inequality. For a constant \( M_0 \geq 0 \), suppose that \( f(t) \geq 0, t \mapsto f(t) \) is continuous, and

\[
f(t) \leq M_0 + C t P(f(t)),
\]

where \( P \) denotes a polynomial function, and \( C \) is a generic constant. Then for \( t \) taken sufficiently small, we have the bound (cf. \([3, 4]\))

\[
f(t) \leq 2 M_0.
\]

For integers \( k \geq 0 \) and a smooth, open domain \( \Omega \) of \( \mathbb{R}^3 \), we define the Sobolev space \( H^k(\Omega) \) \((H^k(\Omega; \mathbb{R}^3))\) to be the completion of \( C^\infty(\Omega) \) \((C^\infty(\Omega; \mathbb{R}^3))\) in the norm

\[
\|u\|_k := \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u(x)|^2 \, dx \right)^{1/2},
\]

for a multi-index \( \alpha \in \mathbb{Z}^3_+ \), with the standard convention \(|\alpha| = \alpha_1 + \alpha_2 + \alpha_3\). For real numbers \( s \geq 0 \), the Sobolev spaces \( H^s(\Omega) \) and the norms \( \| \cdot \|_s \) are defined by interpolation. We will write \( H^s(\Omega) \) instead of \( H^s(\Omega; \mathbb{R}^3) \) for vector-valued functions. In the case that \( s \geq 3 \), the above definition also holds for domains \( \Omega \) of class \( H^3 \).

Our analysis will often make use of the following subspace of \( H^1(\Omega) \):

\[
H^1_0 = \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma, (x_1, x_2) \mapsto u(x_1, x_2) \text{ is periodic} \}.
\]
where, as usual, the vanishing of $u$ on $\Gamma$ is understood in the sense of trace.

For functions $u \in H^k(\Gamma)$, $k \geq 0$, we set
\[
|u|_k := \left( \sum_{|\alpha| \leq k} \int_\Omega |\nabla^\alpha u(x)|^2 \, dx \right)^{1/2},
\]
for a multi-index $\alpha \in \mathbb{Z}^d_+$. For real $s \geq 0$, the Hilbert space $H^s(\Gamma)$ and the boundary norm $|\cdot|_s$ is defined by interpolation. The negative-order Sobolev spaces $H^{-s}(\Gamma)$ are defined via duality: for real $s \geq 0$,
\[
H^{-s}(\Gamma) := [H^s(\Gamma)]'.
\]

The derivative loss inherent to this degenerate problem is a consequence of the weighted embedding we now describe.

Using $d$ to denote the distance function to the boundary $\Gamma_1$, i.e., $d(x) = \text{dist} (x, \Gamma_1)$, and letting $p = 1$ or $2$, the weighted Sobolev space $H_{dp}^1(\Omega)$, with norm given by
\[
\int_{\Omega} d(x)^p \left( |F(x)|^2 + |\nabla F(x)|^2 \right) dx
\]
for any $F \in H_{dp}^1(\Omega)$, satisfies the following embedding:
\[
H_{dp}^1(\Omega) \hookrightarrow H^{1-\frac{2}{p}}(\Omega);
\]
that is, there is a constant $C > 0$ depending only on $\Omega$ and $p$, such that
\[
\|F\|^2_{1,p/2} \leq C \int_{\Omega} d(x)^p \left( |F(x)|^2 + |\nabla F(x)|^2 \right) dx. \tag{2.4}
\]

See, for example, Section 8.8 in Kufner \[10\]. From this embedding relation and (1.7), we obtain
\[
\|F\|^2_{0,0} \leq C \int_{\Omega} \rho_0^2 \left( |F(x)|^2 + |\nabla F(x)|^2 \right) dx. \tag{2.5}
\]

2.2. Higher-order Hardy-type inequality and Hodge-type elliptic estimates. We will make fundamental use of the following generalization of the well-known Hardy inequality to higher-order derivatives, see \[6, Lemma 3.1\]:

**Lemma 2.1** (Higher-order Hardy-type inequality). Let $s \geq 1$ be a given integer, $\Omega$ and $d(x)$ be defined as above, and suppose that
\[
u u \in H^s(\Omega) \cap \dot{H}^1_0(\Omega),
\]
then \(\frac{\nu}{d} \in H^{s-1}(\Omega)\) and
\[
\left\| \frac{\nu}{d} \right\|_{s-1} \leq C \|u\|_s. \tag{2.6}
\]

From this lemma, we can obtain

**Corollary 2.2.** Assume that $p \in [2, \infty]$ and $\rho_0$ satisfies the conditions (1.6) and (1.7), then
\[
\left\| \frac{g}{\rho_0} \right\|_{L^p(\Omega)} \leq C \|g\|_{\left[ \frac{1}{2} - \frac{1}{p} \right] + 1}, \quad \forall g \in H^{\left[ \frac{1}{2} - \frac{1}{p} \right] + 1}(\Omega) \cap \dot{H}^1_0(\Omega), \tag{2.7}
\]
where $\left[ \cdot \right]$ is the ceiling function defined by $\left[ p \right] = \min\{n \in \mathbb{Z} : n \geq p\}$.

**Proof.** By (1.7) and noticing that $d(x) = \text{dist} (x, \Gamma_1)$, we see that
\[
\frac{1}{\rho_0(x)} \leq C \frac{1}{d(x)} \quad \text{for } x \text{ near } \Gamma_1. \tag{2.8}
\]
Since \( \rho_0 > 0 \) in \( \Omega \) and (1.6), for \( x \in \Omega \), we have \(|\partial_3 \rho_0| > \varepsilon \) for some \( \varepsilon > 0 \), then \( \rho_0(x) = |\partial_3 \rho_0(x + \theta(1 - x))||1 - \mathbf{x}_3| > \varepsilon d(x) \) for \( x \in \Omega \) far away from \( \Gamma_1 \) with some \( \theta \in [0, 1] \). Thus, combining with (2.8), we have

\[
\frac{1}{\rho_0(x)} \leq \frac{C}{d(x)} \quad \text{for} \quad x \in \Omega. \tag{2.9}
\]

It follows, from Sobolev’s embedding inequality and Lemma 2.1, that for \( p \in [2, \infty] \)

\[
\left\| \frac{g}{\rho_0} \right\|_{L^p(\Omega)} \leq C \left\| \frac{g}{d} \right\|_{L^p(\Omega)} \leq C \left\| \frac{g}{d} \right\|_{[3(1/2-1/p)]+1},
\]

if \( g \in H^{3(1/2-1/p)+1}(\Omega) \cap H^1_0(\Omega) \).

The normal trace theorem provides the existence of the normal trace \( w \cdot N \) of a velocity field \( w \in L^2(\Omega) \) with \( \text{div} \ w \in L^2(\Omega) \) (see, e.g., [13]). For our purposes, the following form is most useful (see, e.g., [1]):

**Lemma 2.3** (Normal trace theorem). Let \( w \) be a vector field defined on \( \Omega \) such that \( \overline{\partial} w \in L^2(\Omega) \) and \( \text{div} \ w \in L^2(\Omega) \), and let \( N \) denote the outward unit normal vector to \( \Gamma \). Then the normal trace \( \overline{\partial} w \cdot N \) exists in \( H^{-1/2}(\Gamma) \) with the estimate

\[
|\overline{\partial} w \cdot N|_{-1/2} \leq C \left[ \|\overline{\partial} w\|_{L^2(\Omega)}^2 + \|\text{div} \ w\|_{L^2(\Omega)}^2 \right], \tag{2.10}
\]

for some constant \( C \) independent of \( w \).

**Lemma 2.4** (Tangential trace theorem). Let \( \overline{\partial} w \in L^2(\Omega) \) so that \( \text{curl} \ w \in L^2(\Omega) \), and let \( T_1, T_2 \) denote the unit tangent vectors on \( \Gamma \), so that any vector field \( u \) on \( \Gamma \) can be uniquely written as \( u^\alpha T_\alpha \). Then

\[
|\overline{\partial} w \cdot T_\alpha|_{-1/2} \leq C \left[ \|\overline{\partial} w\|_{L^2(\Omega)}^2 + \|\text{curl} \ w\|_{L^2(\Omega)}^2 \right], \quad \alpha = 1, 2 \tag{2.11}
\]

for some constant \( C \) independent of \( w \).

Combining (2.10) and (2.11), we have

\[
|\overline{\partial} w|_{-1/2} \leq C \left[ \|\overline{\partial} w\|_{L^2(\Omega)} + \|\text{div} \ w\|_{L^2(\Omega)} + \|\text{curl} \ w\|_{L^2(\Omega)} \right] \tag{2.12}
\]

for some constant \( C \) independent of \( w \).

The construction of our higher-order energy function is based on the following Hodge-type elliptic estimate (see, e.g., [3]):

**Lemma 2.5.** For an \( H^r \) domain \( \Omega \), \( r \geq 3 \), if \( F \in L^2(\Omega; \mathbb{R}^3) \) with \( \text{curl} F \in H^{s-1}(\Omega) \), \( \text{div} F \in H^{s-1}(\Omega) \), and \( F \cdot N|_\Gamma \in H^{s-1/2}(\Gamma) \) for \( 1 \leq s \leq r \), then there exists a constant \( \tilde{C} > 0 \) depending only on \( \Omega \) such that

\[
\|F\|_s \leq \tilde{C} \left( \|F\|_0 + \|\text{curl} \ F\|_{s-1} + \|\text{div} F\|_{s-1} + |\overline{\partial} F \cdot N|_{s-3/2} \right),
\]

\[
\|F\|_s \leq \tilde{C} \left( \|F\|_0 + \|\text{curl} \ F\|_{s-1} + \|\text{div} F\|_{s-1} + \sum_{\alpha=1}^2 |\overline{\partial} F \cdot T_\alpha|_{s-3/2} \right),
\]

where \( N \) denotes the outward unit normal to \( \Gamma \) and \( T_\alpha \) are tangent vectors for \( \alpha = 1, 2 \).
2.3. **Properties of \( J, A \) and \( a \).** From the derivative formula of matrices and determinants, we have

\[
A^k_i - A^k_i \eta^{l_j}_s A^j_i,
\]

\[
A^k_i = \partial_j A^k_i = -A^k_i \eta^{\ell_j}_s A^j_i,
\]

\[
J_s = J A^k_i \eta^{l_j}_s = a^j_i \eta^{l_j}_s.
\]  

(2.13) \hspace{2cm} (2.14) \hspace{2cm} (2.15)

It follows from \( a = JA, \) (2.13) and (2.15) that the columns of every adjoint matrix are divergence-free, i.e., the Piola identity,

\[
a^k_{i,k} = 0,
\]

(2.16)

which will play a vital role in our energy estimates. We also have

\[
a^k_{i,s} = J \eta^{l_j}_s (A^k_i A^j_s - A^k_i A^j_s) = J^{-1} \eta^{l_j}_s (a^k_i a^j_s - a^k_i a^j_s),
\]

\[
a^k_{ii} = J \eta^{l_j}_s (A^k_i A^j_s - A^k_i A^j_s) = J^{-1} \eta^{l_j}_s (a^k_i a^j_s - a^k_i a^j_s).
\]  

(2.17) \hspace{2cm} (2.18)

3. **A MIXED SPACE-TIME INTERPOLATION INEQUALITY**

In this section, we prove a useful mixed space-time interpolation inequality which will play a vital role in our energy estimates.

**Proposition 3.1** (Mixed interpolation inequality). Let \( F(t,x) \) be a scalar or vector-valued function for \( t \in [0,T], \ T > 0 \) and \( x \in \Omega \subset \mathbb{R}^3 \). Assume that \( F_i(0,\cdot) \in L^3(\Omega), \ F \in L^\infty([0,T];L^6(\Omega)) \) and \( F_{lt} \in L^\infty([0,T];L^2(\Omega)) \), then we have

\[
\| F_t \|^2_{L^3([0,T] \times \Omega)} \leq C T^{2/3} \left[ \| F_i(0) \|^2_{L^3(\Omega)} + \sup_{[0,T]} \| F(t) \|_{L^6(\Omega)} \| F_{lt}(t) \|_{L^2(\Omega)} \right],
\]

(3.1)

where \( C \) is a constant independent of \( T, \Omega \) and \( F \).

**Proof.** Notice that

\[
2|F_i|(|\partial_t|F_i|F_i| = |\partial_t(|F_i|^2|F_i = 2(F_i \cdot F_{lt}) F_i \leq 2|F_i|^2|F_{lt}|
\]

implies that \( |\partial_t(|F_i|F_i| \leq |F_i||F_{lt}| \). Then, by the Fubini theorem, integration by parts with respect to time, the fundamental theorem of calculus, the Hölder inequality and the Minkowski inequality, we have

\[
\| F_t \|^3_{L^3([0,T] \times \Omega)} = \int_0^T \int_\Omega |F_t|^3 \, dx \, dt = \int_0^T \int_\Omega \, |F_t| \cdot F_t \, dt \, dx
\]

\[
= \int_\Omega \left( \int_0^T \partial_t(|F_t|F_t) \, dt \right) \cdot F(t) \, dx + \int_\Omega \, |F(t)(0)|F(t) \cdot \left( \int_0^T F_t \, dt \right) \, dx
\]

\[
- \int_\Omega \left( \int_0^T |F_t|F_{lt} \cdot F \, dx \, dt \right) - \int_\Omega \left( \int_0^T \partial_t(|F_t|F_t) \cdot F \, dt \right) \, dx
\]

\[
\leq 2 \int_\Omega \left( \int_0^T \partial_t(|F_t|F_t) \, dt \right) \cdot F(t) \, dx + \int_\Omega \, |F(t)(0)|^2 \left( \int_0^T F_t \, dt \right) \, dx + 2 \int_\Omega \int_0^T |F_t|F_{lt}||F| \, dx \, dt
\]

\[
\leq C \| F(T) \|^2_{L^3(\Omega)} \int_0^T \| F_t \|_{L^6(\Omega)} \, dt + \| F(t)(0) \|^2_{L^3(\Omega)} \int_0^T \| F_t \|_{L^6(\Omega)} \, dt
\]

\[
+ C \| F_t \|^2_{L^3([0,T] \times \Omega)} \| F_{lt} \|_{L^2([0,T] \times \Omega)} \| F \|^2_{L^6([0,T] \times \Omega)} \| F \|^2_{L^6([0,T] \times \Omega)}
\]

\[
\leq C T^{2/3} \| F_t \|^3_{L^3([0,T] \times \Omega)} \left[ \| F(t)(0) \|^2_{L^3(\Omega)} + \sup_{[0,T]} \| F(t) \|_{L^6(\Omega)} \| F_{lt} \|_{L^2(\Omega)} \right],
\]
which implies the desired inequality by eliminating \( \|F_t\|_{L^1([0,T] \times \Omega)} \) from both sides of the inequality.

\[ \square \]

4. A PRIORI ASSUMPTION AND THE ZERO-TH ORDER ENERGY ESTIMATES

We assume that we have smooth solutions \( \eta \) on a time interval \([0,T]\), and that for all such solutions, the time \( T > 0 \) is taken sufficiently small so that for \( t \in [0,T] \),

\[ \frac{1}{2} \leq J(t) \leq \frac{3}{2}. \tag{4.1} \]

Once we establish the a priori bounds, we can ensure that our solution verifies the assumption (4.1) by means of the fundamental theorem of calculus. Then, by (1.9), Sobolev’s embedding \( H^2(\Omega) \subset L^\infty(\Omega) \), we have for \( t \in [0,T] \),

\[ \|\eta(t)\|_{L^\infty(\Omega)} \leq C\|\eta(t)\|_2, \tag{4.2} \]

\[ \|\alpha(t)\|_{L^\infty(\Omega)} \leq C\|\nabla \eta(t)\|_{L^2(\Omega)}^2 \leq C\|\eta(t)\|_3^2. \tag{4.3} \]

It follows from \( a = JA \) and (4.1) that

\[ \|A(t)\|_{L^\infty(\Omega)} \leq \|\eta^{-1}(t)\alpha(t)\|_{L^\infty(\Omega)} \leq C\|\eta(t)\|_3^2. \tag{4.4} \]

Now, we prove the following zero-th order energy estimates.

**Proposition 4.1.** It holds for \( r \geq 4 \)

\[ \sup_{[0,T]} \left[ \|\rho_0^{1/2}v\|_0^2 + \|\rho_0J^{-1/2}\|_0^2 \right] \leq M_0 + CT^P(\sup_{[0,T]} E_r(t)). \]

**Proof.** Taking the \( L^2 \)-inner product of (1.13a) with \( v^i \) yields, by integration by parts, (2.16), (1.13b), (1.13c) and (1.11), that

\[ \frac{1}{2} \frac{d}{dt} \int_\Omega \rho_0|v|^2 dx = - \int_\Omega a_i^j \left( \rho_0^2 J^{-2} \right)_{ij} v^i v^j dx = \int_\Omega \rho_0^2 J^{-2} a_i^j v^i v^j dx - \int_{\Gamma_0} a_i^j \rho_0^2 J^{-2} v^i v^j dx \]

\[ = - \frac{d}{dt} \int_\Omega \rho_0^2 J^{-1} dx. \]

It follows, from the integration over \([0,T]\) and by Hölder’s inequality, (4.1) and (1.9), that

\[ \frac{1}{2} \int_\Omega \rho_0|v|^2 dx + \int_\Omega \rho_0^2 J^{-1} dx = \frac{1}{2} \int_\Omega \rho_0|u_0|^2 dx + \int_\Omega \rho_0^2 dx = \frac{1}{2} \|\rho_0^{1/2}u_0\|_0^2 + \|\rho_0\|_0^2, \]

which implies, by Cauchy’s inequality, that

\[ \frac{1}{4} \sup_{[0,T]} \|\rho_0^{1/2}v\|_0^2 + \sup_{[0,T]} \|\rho_0J^{-1/2}\|_0^2 \leq M_0 + CT^P(\sup_{[0,T]} E_r(t)). \]

Thus, we complete the proof. \( \square \)

5. THE CURL ESTIMATES

Taking the Lagrangian curl of (1.13a) yields that

\[ \epsilon_{lji} A_j^s(\rho_0 v^i_t + a_i^k (\rho_0^2 J^{-2})_{,k,})_{,s} = 0, \]

i.e.,

\[ \epsilon_{lji} A_j^s \rho_{0,s} v^i_t + \rho_0 \epsilon_{lji} A_j^s v^i_{s,} + \epsilon_{lji} A_j^s a_i^k (\rho_0^2 J^{-2})_{,k,} + \epsilon_{lji} JA_j^s A_i^k (\rho_0^2 J^{-2})_{,ks} = 0. \]
For the last term, we can interchange the indices $k$ and $s$, $i$ and $j$ to find that it vanishes due to the fact $\epsilon_{lij} = -\epsilon_{lij}$. For the first term, we can use the equation (1.13a). Then using (2.17) for the third term, it follows that

\[-\epsilon_{lj} \frac{\rho_0 s}{\rho^2} A^s_j A^k_l (\rho_0^2 J^{-2})_k + \rho_0 \epsilon_{lijk} A^k_l v^l_t + \epsilon_{lij} A^s_j \eta_{b s k} (A^b_k A^k_l - A^b l A^l k) (\rho_0^2 J^{-2})_k = 0.\]

Interchanged the indices $b$ and $s$, $i$ and $j$, the term $\epsilon_{lijk} A^s_j \eta_{b s k} (A^b_k A^k_l - A^b l A^l k) (\rho_0^2 J^{-2})_k$ vanishes. Hence, from (2.15), we have

\[
-\epsilon_{lijk} \frac{\rho_0 s}{\rho^2} A^s_j A^k_l (\rho_0^2 J^{-2})_k - \epsilon_{lijk} A^s_j \eta_{b s k} A^k_l (\rho_0^2 J^{-2})_k =
\]

\[
= 2\epsilon_{lijk} \frac{\rho_0 s}{\rho^2} A^s_j A^k_l \rho_0 \rho_0 J^{-1} - 2\epsilon_{lijk} \rho_0 \rho_0 J^{-1} A^s_j \eta_{b s k} A^k_l \rho_0 - 2\epsilon_{lijk} A^s_j \eta_{b s k} A^k_l \rho_0 \rho_0 J^{-1} + 2\epsilon_{lijk} A^s_j \eta_{b s k} A^k_l (\rho_0^2 J^{-2})_k.
\]

The first and last terms vanish by interchanging the indices $k$ and $s$, $i$ and $j$. The second and third terms eliminate. Therefore, it yields

\[
\epsilon_{lijk} A^s_j v^l_t = 0, \quad \text{or} \quad \text{curl}_t v_t = 0. \tag{5.1}
\]

We can obtain the following proposition.

**Proposition 5.1.** For all $t \in (0, T]$, we have for $r \geq 4$

\[
\sum_{\ell=0}^{r-1} \|\text{curl} \partial_t^\ell \eta(t)\|_{L^2} + \sum_{\ell=0}^{r-1} \|\rho_0 \partial_t^\ell \eta(t)\|_2^2 \leq M_0 + CTP(\sup_{[0, T]} E_r(t)). \tag{5.2}
\]

**Proof.** From (5.1), we get

\[
\partial_t (\text{curl}_t v) = \epsilon_{lijk} A^s_j v^l_t.
\]

Integrating over $(0, t)$ yields

\[
\text{curl}_t v(t) = \text{curl}_t u_0 + \int_0^t \epsilon_{lijk} A^s_j v^l_t dt', \tag{5.3}
\]

and computing the $r$-th order horizontal derivatives of this relation yields

\[
\text{curl}_t \partial_t^r v(t) = \partial_t^r (\text{curl}_t v(t)) - \epsilon_{lijk} \partial_t^r A^s_j v^l_t = -\epsilon_{lijk} \sum_{k=1}^{r-1} C^s_j \partial_t^{r-k} A^s_j \partial_t^k v^l_t.
\]

Noticing that $\eta_t = v$, we have

\[
\partial_t (\text{curl}_t \partial_t^r \eta) = \text{curl}_t \partial_t^r v + \epsilon_{lijk} A^s_j \partial_t^r \eta^l_t,
\]

\[
= \partial_t^r \text{curl}_t u_0 - \epsilon_{lijk} \partial_t^r A^s_j v^l_t - \epsilon_{lijk} \sum_{k=1}^{r-1} C^s_j \partial_t^{r-k} A^s_j \partial_t^k v^l_t + \int_0^t \epsilon_{lijk} \partial_t^r (A^s_j v^l_t) dt'.
\]

Integrating over $[0, t]$ yields

\[
\text{curl}_t \partial_t^r \eta = t \partial_t^r \text{curl}_t u_0 + \int_0^t \epsilon_{lijk} A^s_j \partial_t^r \eta^l_t A^m_j v^l_t dt' - \sum_{k=1}^{r-1} C^s_j \int_0^t \epsilon_{lijk} \partial_t^{r-k} A^s_j \partial_t^k v^l_t dt' - \int_0^t \epsilon_{lijk} A^s_j \partial_t^r (A^m_p v^p_q A^q_j v^l_t) dt'' dt',
\]

where $m, n = 1, 2, 3$. The proof is complete.
and then by the fundamental theorem of calculus
\[ \text{curl}_\eta \mathbf{\nabla} \mathbf{a} = \text{curl} \mathbf{\nabla} \mathbf{a} + \int_0^t \varepsilon_{ji} A_{ij}^t(t') dt' \mathbf{\nabla} \mathbf{a}^i_{,s}. \] (5.5)

It follows that
\[ \mathbf{\nabla} \text{curl} \eta = t \mathbf{\nabla} \text{curl} u_0 + \int_0^t \varepsilon_{ji} A_{ij}^t \eta^i_{,m} A_{jm}^v \mathbf{v}^j_{,s} dt' - \sum_{k=1}^{r-1} C_k \int_0^t \varepsilon_{ji} \mathbf{\nabla}^{(k)} A_{ij} \mathbf{v}^j_{,s} dt' \]
\[- \int_0^t \varepsilon_{ji} A_{ij}^t \eta^i_{,s} dt' - \int_0^t \varepsilon_{ji} A_{ij}^t(t') dt' \mathbf{\nabla} \mathbf{a}^i_{,s} dt' - \int_0^t \varepsilon_{ji} \mathbf{\nabla} (A_{p} v^p q A_j^q \mathbf{v}^j_{,s}) dt'' dt'. \]

Notice that for \( k = 1 \), integration by parts with respect to time and the fact \( \mathbf{\nabla} \nabla \eta(0) = 0 \) imply
\[ \int_0^t \varepsilon_{ji} \mathbf{\nabla}^{(1)} A_{ij} \mathbf{v}^j_{,s} dt' = - \int_0^t \varepsilon_{ji} \mathbf{\nabla}^{(2)} (A_{p} \mathbf{\nabla} \eta^p q A_j^q) \mathbf{v}^j_{,s} dt' \]
\[= - \int_0^t \varepsilon_{ji} A_{ij} \mathbf{\nabla} \eta^i_{,s} dt' - \sum_{l=0}^{r-3} C_{l-2} \int_0^t \mathbf{\nabla}^{(l+1)} (A_{p} A_j^q) \mathbf{\nabla}^{(l+1)} \eta^p q \mathbf{\nabla} \mathbf{v}^j_{,s} dt' \]
\[= - \varepsilon_{ji} A_{ij} \mathbf{\nabla} \eta^i_{,s} + \int_0^t \varepsilon_{ji} A_{ij} \mathbf{\nabla} \mathbf{v}^j_{,s} dt' + \int_0^t \varepsilon_{ji} A_{ij} \mathbf{\nabla} \mathbf{v}^j_{,s} dt'. \]

Since other terms can be estimated easily, thus, it follows that
\[ \sup_{[0,T]} \int_\Omega |\mathbf{\nabla} \text{curl} \eta|^2 dx \leq M_0 + \delta \sup_{[0,T]} \| E_r(t) + CT \| \] (sup \( E_r(t) \)).

The weighted estimates for the curl of \( \mathbf{\nabla}^{2m} \eta \) can be obtained similarly.

From (5.3), we see that
\[ \text{curl}_\eta \eta = t \text{curl} u_0 + \int_0^t \varepsilon_{ji} A_{ij}^t \eta^i_{,s} dt' + \int_0^t \int_0^t \varepsilon_{ji} A_{ij}^s \mathbf{v}^j_{,s} dt'' dt', \]
and then by the fundamental theorem of calculus,
\[ \text{curl} \eta = t \text{curl} u_0 - \varepsilon_{ji} \int_0^t A_{ij}^t dt' \eta^i_{,s} + \int_0^t \int_0^t \varepsilon_{ji} A_{ij}^s \mathbf{v}^j_{,s} dt'' dt'. \]

It follows that
\[ \| \text{curl} \eta \|^2_{L^2} \leq M_0 + CT \| \sup_{[0,T]} E_r(t) \|, \]
and
\[ \| \mathbf{\nabla}^{r-1} \text{curl} \eta \|^2_{L^2} \leq M_0 + CT \| \sup_{[0,T]} E_r(t) \|, \]
where we have used integration by parts with respect to time for the integrals involving \( \mathbf{\nabla}^{r-1} \mathbf{v} \) in order to control them by \( \| \eta \|_r \) and \( \| \mathbf{v} \|_{r-1} \). Thus, we get
\[ \| \text{curl} \eta \|_{r-1} \leq M_0 + CT \| \sup_{[0,T]} E_r(t) \|. \] (5.6)

From (5.3), we get, by the fundamental theorem of calculus,
\[ \text{curl} \mathbf{v}(t) = \text{curl} u_0 - \varepsilon_{ji} \int_0^t A_{ij}^t dt' \mathbf{v}^j_{,s} + \varepsilon_{ji} \int_0^t A_{ij}^s \mathbf{v}^j_{,s} dt', \]
and, by taking the first order time derivative, 
\[
\text{curl} \, v_i(t) = -\varepsilon_{ji} \int_0^t A_i^j \, dt' \, v_i^j.
\]
Since \( H^{r-2}(\Omega) \) is a multiplicative algebra for \( r \geq 4 \), we can directly estimate the \( H^{r-2}(\Omega) \)-norm of \( \text{curl} \, v_i \) to show that 
\[
\|\text{curl} \, v_i(t)\|_{r-2}^2 \leq M_0 + CTP(\sup_{[0,T]} E_r(t)).
\]

The estimates for \( \text{curl} \, \partial_t^{2m} \eta(t) \) in \( H^{r-1-m}(\Omega) \) for \( 2 \leq m \leq r-1 \) follow from the same arguments. \( \Box \)

6. The Estimates for the Horizontal Derivatives

We have the following estimates.

**Proposition 6.1.** Let \( r \in \{4, 5\} \). For small \( \delta > 0 \) and the constant \( M_0 \) depending on \( 1/\delta \), it holds
\[
\sup_{[0,T]} \left[ \|\rho_0^{1/2} \hat{\sigma}^r(v)(t)\|_{\tilde{H}} + \|\rho_0 \hat{\sigma}^r \nabla \eta(t)\|_{\tilde{H}} + \|\rho_0 \hat{\sigma}^r \text{div} \eta(t)\|_{\tilde{H}} + |\eta^\alpha|_{r-1/2}\right] 
\leq M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)),
\]
where \( \eta^\alpha = \eta \cdot T_\alpha \) for \( \alpha = 1, 2 \).

**Proof.** Letting \( \hat{\sigma}^r \) act on (1.13a), then we have
\[
\sum_{l=0}^r C_r^l \hat{\sigma}^{r-l} \rho_0 \hat{\sigma}^l v_i^l + \sum_{l=0}^r C_r^l \hat{\sigma}^{r-l} a_i^l \hat{\sigma}^l (\rho_0^2 J^{-2})_{i,j} = 0,
\]
where \( C_r^l \) is the binomial coefficient. Taking the \( L^2(\Omega) \)-inner product with \( \hat{\sigma}^r v_i \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho_0 |\hat{\sigma}^r v_i|^2 \, dx + \sum_{m=0}^3 \mathcal{J}_m = \sum_{m=6}^6 \mathcal{J}_m. \quad (6.1)
\]

Here,
\[
\mathcal{J}_1 := \int_\Omega \hat{\sigma}^r a_i^l (\rho_0^2 J^{-2})_{i,j} \hat{\sigma}^r v_i^l \, dx, \quad \mathcal{J}_2 := \int_\Omega a_i^l \left( \hat{\sigma} \rho_0 J^{-2} \right)_{i,j} \hat{\sigma}^r v_i^l \, dx,
\]
\[
\mathcal{J}_3 := \int_\Omega a_i^l \left( \rho_0^2 \hat{\sigma}^r J^{-2} \right)_{i,j} \hat{\sigma}^r v_i^l \, dx, \quad \mathcal{J}_4 := -\sum_{l=0}^{r-1} C_r^l \int_\Omega \hat{\sigma}^{r-l} \rho_0 \hat{\sigma}^l v_i^l \hat{\sigma}^r v_i^l \, dx,
\]
\[
\mathcal{J}_5 := -\sum_{l=1}^{r-1} C_r^l \int_\Omega \hat{\sigma}^{r-l} a_i^l \hat{\sigma}^l (\rho_0^2 J^{-2})_{i,j} \hat{\sigma}^r v_i^l \, dx, \quad \mathcal{J}_6 := -\sum_{l=1}^{r-1} C_r^l \int_\Omega a_i^l \left( \hat{\sigma}^{-l} \rho_0 \hat{\sigma}^2 J^{-2} \right)_{i,j} \hat{\sigma}^r v_i^l \, dx.
\]

**Step 1. Analysis of the integral \( \mathcal{J}_1 \).** We use the identity (2.16) to integrate by parts with respect to \( x_j \) to find that
\[
\mathcal{J}_1 = -\int_\Omega \hat{\sigma}^r a_i^l \rho_0^2 J^{-2} \hat{\sigma}^r v_i^l \, dx + \int_0^t \int_\Omega \hat{\sigma}^r a_i^l \hat{\sigma}^r v_i^l \rho_0^2 J^{-2} \, dx_1 \, dx_2 = -\int_\Omega \hat{\sigma}^r a_i^l \hat{\sigma}^r v_i^l \rho_0^2 J^{-2} \, dx,
\]
due to the boundary conditions (1.13b) and (1.13c).

It follows from (2.17) that
\[
\mathcal{J}_1 = -\int_\Omega \hat{\sigma}^{-1} \left( \hat{\sigma} \eta_{i,k} J^{-1} (a_i^l a_k^j - a_k^l a_i^j) \right) \hat{\sigma}^r v_i^l \rho_0^2 J^{-2} \, dx
\]
\[
= -\int_\Omega \hat{\sigma}^r \eta_{i,k} A_i^k \hat{\sigma}^r v_i^l A_i^l \rho_0^2 J^{-1} \, dx. \quad (6.2)
\]
Thus, we have

\begin{equation}
(6.2) = -\frac{1}{2} \frac{d}{dt} \int_\Omega |\text{div} \vec{\tau} \eta|^2 \rho_0^2 J^{-1} dx + \int_\Omega \vec{\tau} \eta \cdot A_k \partial_i \rho_0^2 J^{-1} dx \\
+ \frac{1}{2} \int_\Omega |\text{div} \vec{\tau} \eta|^2 \rho_0^2 \partial_t J^{-1} dx
\end{equation}

\begin{equation}
= -\frac{1}{2} \frac{d}{dt} \int_\Omega |\text{div} \vec{\tau} \eta|^2 \rho_0^2 J^{-1} dx + \frac{1}{2} \int_\Omega \vec{\tau} \eta \cdot A_k \partial_i (A_i^k A_i^j) \rho_0^2 J^{-1} dx \\
+ \frac{1}{2} \int_\Omega |\text{div} \vec{\tau} \eta|^2 \rho_0^2 \partial_t J^{-1} dx.
\end{equation}

For the integral (6.3), since \( v = \eta \), it holds

\[ \vec{\tau} \eta \cdot A_k \vec{\tau} \eta \cdot A_j = \partial_i (\vec{\tau} \eta \cdot A_k \vec{\tau} \eta \cdot A_j) - \vec{\tau} \eta \cdot A_k \vec{\tau} \eta \cdot A_j - \vec{\tau} \eta \cdot A_k \vec{\tau} \eta \cdot A_j \partial_i (A_i^k A_i^j). \]

It follows from (2.2) that

\begin{equation}
\vec{\tau} \eta \cdot A_k \vec{\tau} \eta \cdot A_j = \frac{1}{2} \partial_i \left[ |\nabla \vec{\tau} \eta|^2 - |\text{curl} \vec{\tau} \eta|^2 \right] - \frac{1}{2} \vec{\tau} \eta \cdot A_k \vec{\tau} \eta \cdot A_j \partial_i (A_i^k A_i^j).
\end{equation}

Thus, we have

\begin{equation}
(6.3) = \frac{1}{2} \frac{d}{dt} \int_\Omega \left[ |\nabla \vec{\tau} \eta|^2 - |\text{curl} \vec{\tau} \eta|^2 \right] \rho_0^2 J^{-1} dx \\
+ \frac{1}{2} \int_\Omega \left[ |\nabla \vec{\tau} \eta|^2 - |\text{curl} \vec{\tau} \eta|^2 \right] \rho_0^2 \text{div} \eta v dx \\
- \frac{1}{2} \int_\Omega \vec{\tau} \eta \cdot A_k \vec{\tau} \eta \cdot A_j \partial_i (A_i^k A_i^j) \rho_0^2 J^{-1} dx.
\end{equation}

It follows that

\begin{equation}
\int_0^T \mathcal{J}_1 dt = \frac{1}{2} \int_\Omega \left[ |\nabla \vec{\tau} \eta(T)|^2 - |\text{div} \vec{\tau} \eta(T)|^2 - |\text{curl} \vec{\tau} \eta(T)|^2 \right] \rho_0^2 J^{-1}(T) dx \\
+ \frac{2}{2} \int_0^T \int_\Omega \left[ |\nabla \vec{\tau} \eta|^2 - |\text{div} \vec{\tau} \eta|^2 - |\text{curl} \vec{\tau} \eta|^2 \right] \rho_0^2 J^{-1} \text{div} \eta v dx dt \\
- \frac{1}{2} \int_0^T \int_\Omega \vec{\tau} \eta \cdot A_k \vec{\tau} \eta \cdot A_j \partial_i (A_i^k A_i^j) \rho_0^2 J^{-1} dx dt + \int_0^T (6.4) dt.
\end{equation}

It is clear that

\[ \int_0^T \int_\Omega \left[ |\nabla \vec{\tau} \eta|^2 - |\text{curl} \vec{\tau} \eta|^2 \right] \rho_0^2 J^{-1} \text{div} \eta v dx dt \leq C \sup_{[0,T]} \| \rho_0 \vec{\tau} \nabla \eta \|_0^2 \| v \|_3 \| \eta \|_3^4 \leq C \mathcal{P} \left( \sup_{[0,T]} E_r(t) \right), \]

and

\[ \int_0^T \int_\Omega \left| \vec{\tau} \eta \cdot A_k \vec{\tau} \eta \cdot A_j \partial_i (A_i^k A_i^j) \rho_0^2 J^{-1} dx dt \right| \leq C \sup_{[0,T]} \| \rho_0 \vec{\tau} \nabla \eta \|_0^2 \| v \|_3 \| \eta \|_3^3 \leq C \mathcal{P} \left( \sup_{[0,T]} E_r(t) \right). \]
Now, we analyze the integral $\int_0^T (6.4) dt$. We will use integration by parts with respect to time for the cases $s = 1$ and $s = 2$, while we have to use integration by parts with respect to spatial variables for the case $s = r - 1$.

**Case 1: $s = 1$.** From integration by parts with respect to time, we get

$$-(r - 1) \int_0^T \int_\Omega \overline{\sigma}^{r-1} \eta^i_j \partial_t (a_i^k A^l_j - A_i^l a_j^k) \overline{\sigma}^r v^j_r dx dt$$

$$\leq -(r - 1) \int_0^T \int_\Omega \overline{\sigma}^{r-1} \eta^i_j \partial_t \left( a_i^k A^l_j - A_i^l a_j^k \right) \overline{\sigma}^r \eta^i_j \rho_0^2 J^{-2} dx dt$$

$$+ (r - 1) \int_0^T \int_\Omega \overline{\sigma}^{r-1} \eta^i_j \partial_t \left( a_i^k A^l_j - A_i^l a_j^k \right) \overline{\sigma}^r \eta^i_j \rho_0^2 J^{-2} dx dt$$

$$+ (r - 1) \int_0^T \int_\Omega \overline{\sigma}^{r-1} \eta^i_j \partial_t \left( a_i^k A^l_j - A_i^l a_j^k \right) \overline{\sigma}^r \eta^i_j \rho_0^2 \partial_t J^{-2} dx dt$$

(6.6)

(6.7)

(6.8)

(6.9)

(6.10)

It is clear that, by Hölder’s inequality and the fundamental theorem of calculus twice,

$$|(6.7)| \leq C \| \rho_0 \int_0^T \overline{\sigma}^{r-1} \partial_t \nabla \eta dt \|_0 \| \rho_0 \overline{\sigma}^r \nabla \eta (T) \| \| \eta (T) \|_4$$

$$\leq C T \left( \sup_{[0,T]} \left( \rho_0 \int_0^T \overline{\sigma}^{r-1} \partial_t^2 \nabla \eta dt \right) + \| \rho_0 \overline{\sigma}^{r-1} \partial_t \nabla \eta (0) \| \right) \| \rho_0 \overline{\sigma}^r \nabla \eta (T) \| \| \eta (T) \|_4$$

$$\leq C T^2 \sup_{[0,T]} \left( \| \rho_0 \overline{\sigma}^{r-1} \partial_t^2 \nabla \eta \|_0 + \sum_{i=1}^2 \| \rho_0 \overline{\sigma}^{r-1} \partial_t^i \nabla \eta (0) \| \right) \| \rho_0 \overline{\sigma}^r \nabla \eta (T) \| \| \eta (T) \|_4$$

$$\leq M_0 + C T^2 P \left( \sup_{[0,T]} E_r (t) \right),$$

(6.11)

(6.12)

and

$$|(6.10)| \leq C T \| \rho_0 \|_2 \sup_{[0,T]} \| \overline{\sigma}^{r-1} \nabla \eta \|_0 P \left( \| \eta \|_4, \| \nabla v \|_2, \| \overline{\sigma}^r \nabla \eta \|_0 \right) \leq M_0 + C T^2 P \left( \sup_{[0,T]} E_r (t) \right).$$

We can rewrite (6.9) as, for $\beta \in \{1, 2\}$

$$-3 \int_0^T \int_\Omega \overline{\sigma}^{r-1} \eta^i_j \partial_t \left( a_i^3 A^l_j - A_i^l a_j^3 \right) J^{-2} \overline{\sigma}^r \eta^i_j \rho_0^2 dx dt$$

$$+ 3 \int_0^T \int_\Omega \overline{\sigma}^{r-1} \eta^i_j \partial_t \left( a_i^\beta A^l_j - A_i^l a_j^\beta \right) J^{-2} \overline{\sigma}^r \eta^i_j \rho_0^2 dx dt$$

(6.11)

(6.12)

Obviously, we have from the Hölder inequality and Sobolev’s embedding theorem that

$$|(6.11)| \leq C T \sup_{[0,T]} \| \rho_0 \overline{\sigma}^r \eta \|_1 P \left( \| \eta \|_4, \| \nabla \eta \|_1 \right) \| \rho_0 \overline{\sigma}^r \nabla \eta \|_0$$

$$\leq M_0 + C T^2 P \left( \sup_{[0,T]} E_r (t) \right).$$
\[
\leq CT \sup_{[0,T]} \left( \| \rho_0 \|_3 \| \eta \|_r + \| \rho_0 \mathcal{D}^r \nabla \eta \|_0 \right) P(\| \eta \|_4, \| v \|_3, \| \mathcal{D} \nabla v \|_1) \| \rho_0 \mathcal{D}^r \nabla \eta \|_0 \\
\leq M_0 + CTP(\sup_{[0,T]} E_r(t)).
\]

By integration by parts, it yields

\begin{equation}
(6.12) = -3 \int_0^T \int_\Omega \mathcal{D}^{r-1} \eta_1 \beta \partial_i \mathcal{D} \left( A_i^j \partial_j A_j^k - A_i^k \partial_j A_j^j \right) \eta^j J^{-2} \rho_0^2 dx dt \\
-3 \int_0^T \int_\Omega \mathcal{D}^{r-1} \eta_1 \beta \partial_i \mathcal{D} \left( A_i^j \partial_j A_j^k - A_i^k \partial_j A_j^l \right) \eta^j J^{-2} \rho_0^2 dx dt \\
-3 \int_0^T \int_\Omega \mathcal{D}^{r-1} \eta_1 \beta \partial_i \mathcal{D} \left( \partial_i A_i^k - A_i^k \partial_i A_i^l \right) \eta^j J^{-2} \rho_0^2 dx dt.
\end{equation}

It is easy to see that

\[
|(6.13)| \leq CT \sup_{[0,T]} \| \rho_0 \mathcal{D}^r \nabla \eta \|_0 \| \mathcal{D} \nabla v \|_1 P(\| \eta \|_4) \| \rho_0 \mathcal{D}^r \eta \|_1 \leq M_0 + CTP(\sup_{[0,T]} E_r(t)),
\]

and

\[
(6.15) \leq CT \sup_{[0,T]} \| \mathcal{D}^{r-1} \nabla \eta \|_0 \| \mathcal{D} \nabla v \|_1 P(\| \eta \|_4) \| \rho_0 \mathcal{D}^r \eta \|_1 \left( \| \rho_0 \|_{L^\infty(\Omega)} + \| \mathcal{D} \rho_0 \|_{L^\infty(\Omega)} \right)
\]

\[
\leq M_0 + CTP(\sup_{[0,T]} E_r(t)).
\]

In order to estimate (6.14), we first consider the estimate of the \( \| D^{r-1} v \|_{L^3([0,T] \times \Omega)} \) where \( D^{r-1} \) denotes all the derivatives \( \partial^\theta \) for the multi-index \( \theta = (\theta_1, \theta_2, \theta_3) \) and \( 0 \leq |\theta| \leq r - 1 \). By Proposition 3.1 with \( F = D^{r-1} \eta \) and the Sobolev embedding theorem, we have

\[
\| D^{r-1} v \|_{L^3([0,T] \times \Omega)}^2 \leq CT^{2/3} \left[ \| D^{r-1} v(0) \|_{L^3(\Omega)} + \sup_{[0,T]} \| D^{r-1} v \|_{L^2(\Omega)} \| D^{r-1} \eta \|_{L^6(\Omega)} \right]
\]

\[
\leq M_0 + CT^{2/3} \sup_{[0,T]} \| v_r \|_{r-1} \| \eta \|_r
\]

\[
\leq M_0 + CT^{2/3} P(\sup_{[0,T]} E_r(t)).
\]

Thus, we obtain

\[
\| D^{r-1} v \|_{L^3([0,T] \times \Omega)}^2 \leq M_0 + CT^{2/3} P(\sup_{[0,T]} E_r(t)).
\]

By the Hölder inequality, the Sobolev embedding theorem, the Cauchy inequality and (6.16), we easily get

\[
|(6.14)| \leq CT^{2/3} \| \rho_0 \|_2 \| \mathcal{D}^2 \nabla v \|_{L^3([0,T] \times \Omega)} \sup_{[0,T]} \| \mathcal{D}^3 \nabla \eta \|_0 \| \rho_0 \mathcal{D}^4 \eta \|_1
\]

\[
\leq CT^{1/3} \| \mathcal{D}^2 \nabla v \|_{L^3([0,T] \times \Omega)}^2 + CT \| \rho_0 \|_2^2 \sup_{[0,T]} \| \mathcal{D}^3 \nabla \eta \|_0^2 \| \rho_0 \mathcal{D}^4 \eta \|_1^2
\]

\[
\leq CT^{1/3} \left( M_0 + CT^{2/3} P(\sup_{[0,T]} E_r(t)) \right) + M_0 + CTP(\sup_{[0,T]} E_r(t))
\]

\[
\leq M_0 + CTP(\sup_{[0,T]} E_r(t)).
\]
Hence, we obtain
\[ \| (6.6) \| \leq M_0 + C T P (\sup_{[0,T]} E_r (t)). \]

**Case 2: s = 2.** By integration by parts with respect to time, it yields
\[
-C_{r-1}^2 \int_0^T \int_\Omega \partial_r^{-2} \eta_l^i \kappa \partial_r^2 (A_l^i d_l^i - A_l^i A_l^i) \partial_r^{j} \rho_0^2 J^{-2} dx dt
\]
\[ \leq C_{r-1}^2 \int_0^T \int_\Omega \partial_r^{-2} \eta_l^i \kappa \partial_r^2 (A_l^i d_l^i - A_l^i A_l^i) \partial_r^{j} \rho_0^2 J^{-2} dx dt \]
\[ + C_{r-1}^2 \int_0^T \int_\Omega \partial_r^{-2} \eta_l^i \kappa \partial_r^2 (A_l^i d_l^i - A_l^i A_l^i) \partial_r^{j} \rho_0^2 J^{-2} dx dt \]
\[ + C_{r-1}^2 \int_0^T \int_\Omega \partial_r^{-2} \eta_l^i \kappa \partial_r^2 (A_l^i d_l^i - A_l^i A_l^i) \partial_r^{j} \rho_0^2 J^{-2} dx dt.
\]

Applying the fundamental theorem of calculus yields for a small \( \delta > 0 \)
\[ \| (6.18) \| \leq C \| \rho_0 \|_2 \| \partial_r^{-2} \nabla \eta (T) \|_1 P (\| \eta (T) \|_4) \| \nabla \eta (T) \|_1 \| \rho_0 \partial_r^{-2} \nabla \eta (T) \|_0 \]
\[ \leq C \| \rho_0 \|_2 P (\| \eta (T) \|_4, \| \eta (T) \|_r) \left( \left\| \int_0^T \nabla v dt \right\|_1 + \| \nabla \eta (0) \|_1 \right) \| \rho_0 \partial_r^{-2} \nabla \eta (T) \|_0 \]
\[ \leq M_0 + \delta \sup_{[0,T]} E_r (t) + C T P (\sup_{[0,T]} E_r (t)). \]

Similar to (6.14), we can get
\[ \| (6.19) \| + \| (6.20) \| \leq M_0 + \delta \sup_{[0,T]} E_r (t) + C T P (\sup_{[0,T]} E_r (t)). \]

By an \( L^6 - L^6 - L^2 \) Hölder inequality and the Sobolev embedding theorem, we get
\[ \| (6.21) \| \leq M_0 + C T P (\sup_{[0,T]} E_r (t)). \]

Therefore, we have obtained
\[ \| (6.17) \| \leq M_0 + \delta \sup_{[0,T]} E_r (t) + C T P (\sup_{[0,T]} E_r (t)). \]

**Case 3: s = r - 1.** We write the space-time integral as, for \( \beta \in \{1, 2\} \)
\[
-C_{r-1}^2 \int_0^T \int_\Omega \partial_r^{-2} \eta_l^i \beta \partial_r^{-1} (a_l^i A_l^k - A_l^i a_l^i) \partial_r^{j} \rho_0^2 J^{-2} dx dt
\]
\[ \leq -C_{r-1}^2 \int_0^T \int_\Omega \partial_r^{-2} \eta_l^i \beta \partial_r^{-1} (a_l^i A_l^k - A_l^i a_l^i) \partial_r^{j} \rho_0^2 J^{-2} dx dt \]
\[ -C_{r-1}^2 \int_0^T \int_\Omega \partial_r^{-2} \eta_l^i \beta \partial_r^{-1} (a_l^i A_l^k - A_l^i a_l^i) \partial_r^{j} \rho_0^2 J^{-2} dx dt.
\]

By integration by parts with respect to \( x_\beta \), we have
\[ (6.23) = -C_{r-1}^2 \int_0^T \int_\Omega \partial_r^{-2} \eta_l^i \beta \partial_r^{-1} (a_l^i A_l^k - A_l^i a_l^i) \partial_r^{j} \rho_0^2 J^{-2} dx dt
\]
\[ + C_{r-1}^2 \int_0^T \int_\Omega \partial_r^{-2} \eta_l^i \beta \partial_r^{-1} (a_l^i A_l^k - A_l^i a_l^i) \partial_r^{j} \rho_0^2 J^{-2} dx dt \]
\[ + C_{r-1}^2 \int_0^T \int_\Omega \partial_r^{-2} \eta_l^i \beta \partial_r^{-1} (a_l^i A_l^k - A_l^i a_l^i) \partial_r^{j} \rho_0^2 J^{-2} dx dt.
\]
From (2.17), it follows that

\begin{equation}
(6.25) = \int_0^T \int_\Omega \tilde{\eta}^{i,j,k} \tilde{\sigma}^{-2} \left[ \tilde{\sigma} \eta^p_{-q} J A^p_j \left( A^q_j A_k^q - A_k^q A_j^q \right) \right] \tilde{\sigma} \rho_0^2 J^{-2} dx dt
\end{equation}

\begin{equation}
+ \int_0^T \int_\Omega \tilde{\eta}^{i,j,k} \tilde{\sigma}^{-2} \left[ \tilde{\sigma} \eta^p_{-q} J A^p_j \left( A^q_j A_k^q - A_k^q A_j^q \right) \right] \tilde{\sigma} \rho_0^2 J^{-2} dx dt
\end{equation}

\begin{equation}
+ \int_0^T \int_\Omega \tilde{\eta}^{i,j,k} \tilde{\sigma}^{-2} \left[ \tilde{\sigma} \eta^p_{-q} J A^p_j \left( A^q_j A_k^q - A_k^q A_j^q \right) \right] \tilde{\sigma} \rho_0^2 J^{-2} dx dt.
\end{equation}

We can write

\begin{equation}
(6.30) = \int_0^T \int_\Omega \tilde{\eta}^{i,j,k} \tilde{\sigma}^{-2} \left[ \tilde{\sigma} \eta^p_{-q} J A^p_j \left( A^q_j A_k^q - A_k^q A_j^q \right) \right] \tilde{\sigma} \rho_0^2 J^{-2} dx dt
\end{equation}

\begin{equation}
+ \int_0^T \int_\Omega \tilde{\eta}^{i,j,k} \tilde{\sigma}^{-2} \left[ \tilde{\sigma} \eta^p_{-q} J A^p_j \left( A^q_j A_k^q - A_k^q A_j^q \right) \right] \tilde{\sigma} \rho_0^2 J^{-2} dx dt
\end{equation}

\begin{equation}
+ \int_0^T \int_\Omega \tilde{\eta}^{i,j,k} \tilde{\sigma}^{-2} \left[ \tilde{\sigma} \eta^p_{-q} J A^p_j \left( A^q_j A_k^q - A_k^q A_j^q \right) \right] \tilde{\sigma} \rho_0^2 J^{-2} dx dt.
\end{equation}

It is easy to see that

\begin{equation}
|| (6.33) \leq C T \| \rho_0 \|_{2}^{1/2} \sup_{[0,T]} || \tilde{\sigma}^2 \nabla \eta ||_1 \| \rho_0 \tilde{\sigma} \eta \|_1 \| P ( || \eta ||_3 ) \|| \rho_0^{1/2} \tilde{\sigma}^2 \|_0 \leq M_0 + C T P( \sup E_r(t)),
\end{equation}

and by an $L^6\cdot L^6\cdot L^2$ Hölder inequality and the Sobolev embedding theorem,

\begin{equation}
|| (6.34) \| + || (6.36) \| \leq C T \| \rho_0 \|_2^{3/2} \sup_{[0,T]} P ( || \eta ||_4, || \eta ||_1 ) \| \rho_0^{1/2} \tilde{\sigma}^2 \|_0 \leq M_0 + C T P( \sup E_r(t)).
\end{equation}

By using integration by parts with respect to time, we have

\begin{equation}
(6.35) = \int_\Omega \tilde{\eta}^{i,j,k} \tilde{\sigma}^{-1} \eta^p_{-q} A^p_j \left( A^q_j A_k^q - A_k^q A_j^q \right) \tilde{\sigma} \eta^i \tilde{\rho}_0^2 J^{-1} dx \bigg|_{t=T}
\end{equation}

\begin{equation}
- \int_0^T \int_\Omega \tilde{\sigma} \eta^{i,j,k} \tilde{\sigma}^{-1} \eta^p_{-q} A^p_j \left( A^q_j A_k^q - A_k^q A_j^q \right) \tilde{\sigma} \eta^i \tilde{\rho}_0^2 J^{-1} dx dt
\end{equation}

\begin{equation}
- \int_0^T \int_\Omega \tilde{\sigma} \eta^{i,j,k} \tilde{\sigma}^{-1} \eta^p_{-q} A^p_j \left( A^q_j A_k^q - A_k^q A_j^q \right) \tilde{\sigma} \eta^i \tilde{\rho}_0^2 J^{-1} dx dt
\end{equation}

\begin{equation}
- \int_0^T \int_\Omega \tilde{\sigma} \eta^{i,j,k} \tilde{\sigma}^{-1} \eta^p_{-q} A^p_j \left( A^q_j A_k^q - A_k^q A_j^q \right) \tilde{\sigma} \eta^i \tilde{\rho}_0^2 dx dt.
\end{equation}
Obviously, it yields by using the fundamental theorem of calculus twice

\[(6.37) \leq C \left( \|\rho_0\|_3 \|\eta\|_3 + \left\| \rho_0 \int_0^T \partial^3 \partial_t \nabla \eta \, dt \right\|_0 \right) P\left( \|\eta(T)\|_3, \|\eta(T)\|_r \right) \|\rho_0 \partial^r \eta(T)\|_1 \]

\[\leq C \|\rho_0\|_3 \|\eta\|_3 \]

\[+ CT \left( \|\rho_0 \partial^3 \partial_t \nabla \eta(0)\|_0 + \left\| \rho_0 \int_0^T \partial^3 \partial_t^2 \nabla \eta \, dt \right\|_0 \right) P\left( \|\eta(T)\|_3, \|\eta(T)\|_r \right) \|\rho_0 \partial^r \eta(T)\|_1 \]

\[\leq M_0 + CTP\left( \sup_{[0,T]} E_r(t) \right). \]

By the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus, we get

\[(6.38) \leq CT \sup_{[0,T]} \|\rho_0 \partial^3 \partial_t \eta\|_1 \|\partial^{r-1} \nabla \eta\|_0 \|\rho_0 \partial^r \eta\|_1 P\left( \|\eta\|_3 \right) \]

\[\leq CT \sup_{[0,T]} \left( \|\rho_0\|_3 \|\eta\|_3 + \|\rho_0 \partial^3 \partial_t \nabla \eta(0)\|_0 + \left\| \rho_0 \int_0^T \partial^3 \partial_t^2 \nabla \eta \, dt \right\|_0 \right) \|\rho_0 \partial^r \eta\|_1 P\left( \|\eta\|_3, \|\eta\|_r \right) \]

\[\leq M_0 + CTP\left( \sup_{[0,T]} E_r(t) \right). \]

Similarly, we have

\[(6.39) \leq CT \sup_{[0,T]} \|\partial^3 \eta\|_1 \|\rho_0 \partial^r \partial_t \eta\|_0 \|\rho_0 \partial^r \eta\|_1 P\left( \|\eta\|_3 \right) \leq M_0 + CTP\left( \sup_{[0,T]} E_r(t) \right), \]

and

\[(6.40) \leq CT \sup_{[0,T]} \|\rho_0\|_2 \|\partial^3 \eta\|_1 \|\partial^{r-1} \nabla \eta\|_0 \|\nabla \eta\|_1 \|\rho_0 \partial^r \eta\|_1 P\left( \|\eta\|_3 \right) \leq M_0 + CTP\left( \sup_{[0,T]} E_r(t) \right). \]

We can deal with (6.31) and (6.32) as the same arguments as for (6.30). Thus, we obtain

\[(6.25) \leq M_0 + CTP\left( \sup_{[0,T]} E_r(t) \right). \]

We write

\[(6.26) = \int_0^T \int_0^T \int_0^T \overline{\partial^3 \partial_t \eta} \overline{\partial^{r-1} \eta} \overline{\eta} \overline{\partial^r \eta} \overline{\rho} \overline{\rho} J^{-1} \, dx \, dt \]

\[+ \int_0^T \int_0^T \int_0^T \overline{\partial^3 \partial_t \eta} \overline{\partial^{r-1} \eta} \overline{\eta} \overline{\partial^r \eta} \overline{\rho} \overline{\rho} J^{-1} \, dx \, dt \]

\[+ \int_0^T \int_0^T \int_0^T \overline{\partial^3 \partial_t \eta} \overline{\partial^{r-1} \eta} \overline{\eta} \overline{\partial^r \eta} \overline{\rho} \overline{\rho} J^{-1} \, dx \, dt \]

\[+ \sum_{m=0}^{r-2} C_{r-1}^m \int_0^T \int_0^T \overline{\partial^3 \partial_t \eta} \overline{\partial^{r-1} \eta} \overline{\eta} \overline{\partial^r \eta} J^{-1} \, dx \, dt \]

\[+ \sum_{m=0}^{r-2} C_{r-1}^m \int_0^T \int_0^T \overline{\partial^3 \partial_t \eta} \overline{\partial^{r-1} \eta} \overline{\eta} \overline{\partial^r \eta} J^{-1} \, dx \, dt \]

\[+ \sum_{m=0}^{r-2} C_{r-1}^m \int_0^T \int_0^T \overline{\partial^3 \partial_t \eta} \overline{\partial^{r-1} \eta} \overline{\eta} \overline{\partial^r \eta} J^{-1} \, dx \, dt \]

\[(6.45) \]

\[(6.46) \]

\[(6.47) \]
It is easy to see that

\[ |(6.42)| + |(6.43)| + |(6.44)| \leq CT \parallel \rho \parallel_{2}^{1/2} \sup_{[0,T]} \parallel \eta \parallel_{4} \parallel \rho_{0} \nabla \eta \parallel_{0} P(\parallel \eta \parallel_{3}) \parallel \rho_{0}^{1/2} \nabla v \parallel_{0} \]

\[ \leq M_{0} + CTP(\sup_{E_{r}(t)}). \]

By the Hölder inequality and the Sobolev embedding theorem, we have

\[ |(6.45)| + |(6.46)| + |(6.47)| \leq CT \parallel \rho_{0}^{3/2} \parallel_{2} \sup_{[0,T]} \parallel \rho_{0}^{1/2} \nabla v \parallel_{0} P(\parallel \eta \parallel_{4}, \parallel \eta \parallel_{r}) \]

\[ \leq M_{0} + CTP(\sup_{E_{r}(t)}). \]

Hence,

\[ |(6.26)| \leq M_{0} + CTP(\sup_{E_{r}(t)}). \] (6.48)

By (2.13), we have

\[ (6.27) = \int_{0}^{T} \int_{\Omega} \eta_{t} \nabla v \rho_{0}^{2}J^{-2} \rho_{0}^{3/2} \]

\[ \int_{0}^{T} \int_{\Omega} \eta_{t} \nabla v \rho_{0}^{2}J^{-2} \rho_{0}^{3/2} \]

\[ = \int_{0}^{T} \int_{\Omega} \eta_{t} \nabla v \rho_{0}^{2}J^{-2} \rho_{0}^{3/2} \]

\[ + \sum_{m=0}^{T} C_{m} \int_{0}^{T} \int_{\Omega} \eta_{t} \nabla v \rho_{0}^{2}J^{-2} \rho_{0}^{3/2} \]

\[ \leq M_{0} + CTP(\sup_{E_{r}(t)}), \]

and similar to (6.45)

\[ |(6.50)| \leq M_{0} + CTP(\sup_{E_{r}(t)}). \]

Thus,

\[ |(6.27)| \leq M_{0} + CTP(\sup_{E_{r}(t)}). \] (6.51)

For (6.28) and (6.29), it is easy to have

\[ |(6.28)| + |(6.29)| \leq CT \parallel \rho \parallel_{3/2} \parallel \rho_{0} \parallel_{2}^{3/2} \sup_{[0,T]} \parallel \rho_{0}^{1/2} \nabla v \parallel_{0} P(\parallel \eta \parallel_{4}, \parallel \eta \parallel_{r}) \]

\[ \leq M_{0} + CTP(\sup_{E_{r}(t)}). \]

Hence,

\[ |(6.23)| \leq M_{0} + CTP(\sup_{E_{r}(t)}). \] (6.52)

By integration by parts with respect to time, it holds

\[ (6.24) = - \int \eta_{t} \rho_{0} \rho_{0}^{2}J^{-2} \rho_{0}^{3/2} \]

\[ + \int_{0}^{T} \int \nabla v \rho_{0} \rho_{0}^{2}J^{-2} \rho_{0}^{3/2} \]

\[ \leq M_{0} + CTP(\sup_{E_{r}(t)}). \]
Both (6.58) and (6.59) can be dealt with as the same argument as for (6.57). Thus, we obtain

\[
\begin{align*}
(6.54) & = \int_0^T \int_{\Omega} \nabla \psi \cdot (\nabla \eta \cdot A_\beta (A_3 A_\alpha - A_\alpha A_3)) \nabla \psi_i \cdot \nabla \eta_i \rho J^{-2} dx dt \\
& + \sum_{m=0}^{r-3} C_{m-2} \int_0^T \int_{\Omega} \nabla \psi \cdot (\nabla \eta \cdot A_3 A_\alpha - A_\alpha A_3) \nabla \psi_i \cdot \nabla \eta_i \rho J^{-2} dx dt.
\end{align*}
\]

It is easy to see that by applying the fundamental theorem of calculus three times

\[
(6.60) \leq C \sup_{[0,T]} \| \nabla \psi \|_1 \| \nabla \eta \|_1 \| P(\| \nabla \|_3) \| \rho J^{-2} \leq M_0 + C T P(\sup E_r(t)).
\]

By (2.17) and (2.13), we have

\[
(6.61) \leq C \sup_{[0,T]} \| \nabla \psi \|_1 \| \nabla \eta \|_1 \| P(\| \nabla \|_3) \| \rho J^{-2} \leq M_0 + C T P(\sup E_r(t)).
\]

Both (6.58) and (6.59) can be dealt with as the same argument as for (6.57). Thus, we obtain

\[
(6.54) \leq M_0 + C T P(\sup E_r(t)).
\]

By (2.18) and (2.14), we have

\[
(6.55) = \int_0^T \int_{\Omega} \nabla \psi \cdot (\nabla \eta \cdot A_3 A_\alpha - A_\alpha A_3) \nabla \psi_i \cdot \nabla \eta_i \rho J^{-2} dx dt \\
+ \sum_{m=0}^{r-3} C_{m-2} \int_0^T \int_{\Omega} \nabla \psi \cdot (\nabla \eta \cdot A_3 A_\alpha - A_\alpha A_3) \nabla \psi_i \cdot \nabla \eta_i \rho J^{-2} dx dt.
\]

Obviously, we see that

\[
(6.60) \leq C T \sup_{[0,T]} \| \nabla \psi \|_1 \| \rho \nabla \eta \|_1 \| P(\| \nabla \|_3) \| \rho J^{-2} \leq M_0 + C T P(\sup E_r(t)).
\]

and

\[
(6.61) \leq C \sup_{[0,T]} \| \nabla \psi \|_1 \| \nabla \eta \|_1 \| P(\| \nabla \|_3) \| \rho J^{-2} \leq M_0 + C T P(\sup E_r(t)).
\]

Both (6.58) and (6.59) can be dealt with as the same argument as for (6.57). Thus, we obtain

\[
(6.54) \leq M_0 + C T P(\sup E_r(t)).
\]
\[ + \int_{0}^{T} \int_{\Omega} \overline{\partial} \eta^{\|} \beta \overline{\partial}^{r-1} \left[ v_{\rho \alpha} J A_{i}^{\beta} (A_{p}^{a} A_{p}^{a} - A_{p}^{a} a_{i}) \right] \overline{\partial}^{r} \eta^{\|} \partial_{\rho} J^{2} dx dt. \] 

(6.64)

We write

\[ (6.62) = \int_{0}^{T} \int_{\Omega} \overline{\partial} \eta^{\|} \beta \overline{\partial}^{r-1} v_{\rho \alpha} A_{i}^{\beta} (A_{p}^{a} A_{p}^{a} - A_{p}^{a} a_{i}) \overline{\partial}^{r} \eta^{\|} \partial_{\rho} J^{2} dx dt \]

\[ + \sum_{m=0}^{r-2} C_{m}^{r} \int_{0}^{T} \int_{\Omega} \overline{\partial} \eta^{\|} \beta \overline{\partial}^{r-1-m} \left[ J A_{i}^{\beta} (A_{p}^{a} A_{p}^{a} - A_{p}^{a} a_{i}) \right] \overline{\partial}^{r} \eta^{\|} \partial_{\rho} J^{2} dx dt. \]

(6.65)

Then, by the same arguments as for (6.44) and (6.47), we can estimate (6.65) and (6.66), and then

\[ |(6.55)| \leq M_{0} + CT \left( \sup_{[0,T]} E_{r}(t) \right). \]

It is easy to see that (6.56) has the same bounds. Thus, we obtain the estimates of (6.24) and then of (6.22), i.e.,

\[ |(6.22)| \leq M_{0} + CT \left( \sup_{[0,T]} E_{r}(t) \right). \]

Case 4: \( s = r - 2 \) and \( r = 5 \). Integration by parts with respect to time gives

\[ \int_{0}^{T} \int_{\Omega} \overline{\partial} \eta^{\|} \beta \overline{\partial}^{r-1} \left( J^{-1}(a_{i} a_{i} - a_{i} a_{i}) \right) \overline{\partial}^{r} v^{i} J^{2} dx dt \]

\[ = \int_{0}^{T} \int_{\Omega} \overline{\partial} \eta^{\|} \beta \overline{\partial}^{r-1} \left( J^{-1}(a_{i} a_{i} - a_{i} a_{i}) \right) \overline{\partial}^{r} \eta^{i} J^{2} dx dt \]

\[ - \int_{0}^{T} \int_{\Omega} \overline{\partial} \eta^{\|} \beta \overline{\partial}^{r-1} \left( J^{-1}(a_{i} a_{i} - a_{i} a_{i}) \right) \overline{\partial}^{r} \eta^{i} J^{2} dx dt \]

\[ - \int_{0}^{T} \int_{\Omega} \overline{\partial} \eta^{\|} \beta \overline{\partial}^{r-1} \left( J^{-1}(a_{i} a_{i} - a_{i} a_{i}) \right) \overline{\partial}^{r} \eta^{i} J^{2} dx dt \]

which can be controlled by \( M_{0} + CT \left( \sup_{[0,T]} E_{5}(t) \right) \) from the Hölder inequality and the fundamental theorem of calculus.

Therefore, we obtain

\[ \left| \int_{0}^{T} (6.4) dt \right| \leq M_{0} + \delta \sup_{[0,T]} E_{r}(t) + C T \left( \sup_{[0,T]} E_{r}(t) \right). \]

Step 2. Analysis of the integral \( \mathcal{I}_{2} \). Similar to those of \( \mathcal{I}_{1} \), by (1.13b), (1.13c) and \( \eta_{t} = v \), we have

\[ \mathcal{I}_{2} = - \int_{\Omega} a_{i}^{3} \overline{\partial}^{r} \rho_{0} J^{2} \overline{\partial}^{r} v^{i} dx + \int_{\Gamma_{0}} a_{i}^{3} \overline{\partial}^{r} \rho_{0} J^{2} \overline{\partial}^{r} v^{i} dx \]

Integration by parts shows that

\[ \int_{0}^{T} \mathcal{I}_{2}(t) dt = \int_{0}^{T} \int_{\Omega} \overline{\partial} \rho_{0} (A_{i}^{\beta} J^{-1}) \overline{\partial} \eta \partial_{\rho} J^{2} dx dt - \int_{\Omega} \overline{\partial} \rho_{0} A_{i}^{\beta} J^{-1} \overline{\partial} \eta \partial_{\rho} J^{2} dx \]

\[ = \int_{0}^{T} \int_{\Omega} \overline{\partial} \rho_{0} (A_{i}^{\beta} J^{-1}) \overline{\partial} \eta \partial_{\rho} J^{2} dx dt - \int_{\Omega} \overline{\partial} \rho_{0} \left( \delta_{i}^{\beta} + \int_{0}^{T} (A_{i}^{\beta} J^{-1}) dt \right) \overline{\partial} \eta \partial_{\rho} J^{2} dx, \]
which yields, by Hölder’s inequality and Sobolev’s embedding theorem, that
\[
\left| \int_0^T \mathcal{J}_2(t)\,dt \right| \leq C \left| \frac{\partial^r \rho_0}{\rho_0} \right|_0 \left( T \sup_{[0,T]} \| \rho_0 \partial^r v \|_0 \| v \|_3 \| \eta \|_3^\delta + \| \rho_0 \partial^r \text{div} \eta \|_0 \right)
\leq M_0 + \delta \sup_{[0,T]} E_r(t) + C TP(\sup_{[0,T]} E_r(t)),
\]
where we require \( \rho_0 \in H^{\max(4, r)}(\Omega) \) because, for \( r \leq 5 \),
\[
\left\| \frac{\partial^r \rho_0}{\rho_0} \right\|_0 \leq \sum_{m=0}^r C^m r \left\| \frac{\partial^m \rho_0 \partial^{r-m} \rho_0}{\rho_0} \right\|_0 \leq 2 \sum_{m=0}^{(r-1)/2} C_m \left\| \frac{\partial^m \rho_0 \partial^{r-m} \rho_0}{\rho_0} \right\|_0
\leq C \left\| \partial^r \rho_0 \right\|_0 + C \left\| \partial^r \rho_0 \right\|_2 \left\| \partial^{r-1} \rho_0 \right\|_0 + C \left\| \partial^2 \rho_0 \right\|_0 \left\| \partial^{r-2} \rho_0 \right\|_2
\leq C \left\| \rho_0 \right\|_r + C \left\| \rho_0 \right\|_4 \left\| \rho_0 \right\|_{r-1} + C \left\| \rho_0 \right\|_3 \left\| \rho_0 \right\|_r
\leq C \left\| \rho_0 \right\|_{\max(4, r)},
\]
by the higher order Hardy inequality.

**Step 3. Analysis of the integral \( \mathcal{J}_3 \).** Similar to those of \( \mathcal{J}_1 \), by (1.13b), (1.13c), (2.15) and \( \eta_t = \nu \), we have
\[
\mathcal{J}_3 = - \int_{\Omega} \rho_0^2 \partial^r a_i \partial^r v_{i,j} \,dx + \int_{\Gamma_0} \rho_0^2 \partial^r a_i \partial^r v^i \,dx_1 \,dx_2
= 2 \int_{\Omega} \rho_0^2 \partial^r(J^{-1}a_i) \partial^r v_{i,j} \,dx
= 2 \int_{\Omega} \partial^r J A^i_l \partial^r v_{i,j} \rho_0^2 J^{-2} \,dx
+ 2 \sum_{s=0}^{r-2} C_{r-1} \int_{\Omega} \partial^{r-1-s} J^{-3} \partial^{s+1} A^i_l \partial^r v_{i,j} \rho_0^2 \,dx.
\]
Due to \( \nu = \eta_r \), we get
\[
(6.68) = 2 \int_{\Omega} \partial^r \eta^k_{i,j} A^i_k \partial^r v_{i,j} \rho_0^2 J^{-1} \,dx
+ 2 \sum_{s=0}^{r-2} C_{r-1} \int_{\Omega} \partial^{r-1-s} a^i_k \partial^{s+1} \eta^k_{i,j} \partial^r v_{i,j} A^i_l \rho_0^2 J^{-2} \,dx
= \frac{d}{dt} \int_{\Omega} \partial^r \eta^k_{i,j} A^i_k \partial^r \eta^i_{j,l} A^l_j \rho_0^2 J^{-1} \,dx + (6.71)
+ \int_{\Omega} \partial^r \eta^k_{i,j} \partial^r \eta^i_{j,l} A^i_k A^l_j \text{div} \eta \rho_0^2 J^{-1} \,dx
- \int_{\Omega} \partial^r \eta^k_{i,j} \partial^r \eta^i_{j,l} (A^i_k A^l_j) \rho_0^2 J^{-1} \,dx.
\]
Noticing that \( \partial^r \text{div} \eta(0) = 0 \), integrating over \([0, T]\) yields
\[
\int_0^T (6.68) \,dt' = \int_{\Omega} [\text{div} \eta \partial^r \eta(T)]^2 \rho_0^2 J^{-1}(T) \,dx + \int_0^T [(6.72) + (6.73) + (6.71)] \,dt.
\]
By Hölder’s inequality and Sobolev’s embedding theorem, we obtain
\[
\left| \int_0^T (6.72) + (6.73) \,dt \right| \leq C T \sup_{[0,T]} \| \rho_0 \partial^r v \|_0^2 \| v \|_3 \| \eta \|_3^\delta \leq C TP(\sup_{[0,T]} E_r(t)).
\]
By integration by parts, we have
\[
\int_0^T (6.71) = 2 \sum_{s=0}^{r-2} C_{r-1}^s \int_0^T \int_\Omega \tilde{\sigma}^{r-1-s} \tilde{a}_k^r \tilde{d}^{s+1} \eta^{k,j} \tilde{\sigma} \tilde{v}^{i,j} \rho_0^2 J^{-2} A_i^l dxdt
\]
\[
= -2 \sum_{s=0}^{r-2} C_{r-1}^s \int_0^T \int_\Omega \tilde{\sigma}^{r-s} \tilde{a}_k^r \tilde{d}^{s+1} \eta^{k,j} \tilde{\sigma}^{r-1} \tilde{v}^{i,j} \rho_0^2 J^{-2} A_i^l dxdt
\]
\[
-2 \sum_{s=0}^{r-2} C_{r-1}^s \int_0^T \int_\Omega \tilde{\sigma}^{r-1-s} \tilde{a}_k^r \tilde{d}^{s+1} \eta^{k,j} \tilde{\sigma}^{r-1} \tilde{v}^{i,j} \rho_0^2 J^{-2} A_i^l dxdt
\]
\[
-2 \sum_{s=0}^{r-2} C_{r-1}^s \int_0^T \int_\Omega \tilde{\sigma}^{r-1-s} \tilde{a}_k^r \tilde{d}^{s+1} \eta^{k,j} \tilde{\sigma}^{r-1} \tilde{v}^{i,j} \rho_0^2 J^{-2} A_i^l dxdt.
\]
(6.74)
(6.75)
(6.76)

We first consider (6.74) and split it into four cases.

**Case 1:** \( s = 0 \). By an \( L^2 - L^\infty \) Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus for the norm \( \| \rho_0 \tilde{\sigma}^{-1} \nabla v \|_0 \), we can easily get
\[
\left| 2 \int_0^T \int_\Omega \tilde{\sigma}^r \tilde{a}_k^r \tilde{\eta}^{k,j} \tilde{\sigma}^{r-1} \tilde{v}^{i,j} \rho_0^2 J^{-2} A_i^l dxdt \right| \leq M_0 + CTP(\sup_{[0,T]} E_r(t)).
\]

**Case 2:** \( s = 1 \). Integration by parts yields
\[
-2 C_{r-1}^1 \int_0^T \int_\Omega \tilde{\sigma}^{r-1} \tilde{a}_k^r \tilde{d}^{2} \eta^{k,j} \tilde{\sigma}^{r-1} \tilde{v}^{i,j} \rho_0^2 J^{-2} A_i^l dxdt
\]
\[
= 2 C_{r-1}^1 \int_0^T \int_\Omega \tilde{\sigma} \tilde{a}_k^r \tilde{d} \eta^{k,j} \tilde{\sigma}^{r-2} \tilde{v}^{i,j} \rho_0^2 J^{-2} A_i^l dxdt
\]
\[
+ 2 C_{r-1}^1 \int_0^T \int_\Omega \tilde{\sigma}^{r-1} \tilde{a}_k^r \tilde{d} \eta^{k,j} \tilde{\sigma}^{r-2} \tilde{v}^{i,j} \rho_0^2 J^{-2} A_i^l dxdt
\]
\[
+ 2 C_{r-1}^1 \int_0^T \int_\Omega \tilde{\sigma}^{r-1} \tilde{a}_k^r \tilde{d} \eta^{k,j} \tilde{\sigma}^{r-2} \tilde{v}^{i,j} \rho_0^2 J^{-2} A_i^l dxdt.
\]
(6.77)
(6.78)
(6.79)
(6.80)

By an \( L^2 - L^6 \) Hölder inequality, (6.16) and the Sobolev embedding theorem, we get
\[
|(6.78)| + |(6.80)| \leq M_0 + CTP(\sup_{[0,T]} E_r(t)).
\]

By using (2.17), it holds
\[
(6.79) = 2 C_{r-1}^1 \int_0^T \int_\Omega \tilde{\sigma}^{r-2} \eta^{p,q} J(A_k^l A_p^q - A_k^q A_p^l) \tilde{\sigma}^3 \eta^{k,j} \tilde{\sigma}^{r-2} \tilde{v}^{i,j} \rho_0^2 J^{-2} A_i^l dxdt
\]
\[
= 2 C_{r-1}^1 \int_0^T \int_\Omega \tilde{\sigma}^{r-1} \eta^{p,\beta} (A_k^l A_p^\beta - A_k^\beta A_p^l) \tilde{\sigma}^3 \eta^{k,j} \tilde{\sigma}^{r-2} \tilde{v}^{i,j} \rho_0^2 J^{-1} A_i^l dxdt
\]
\[
+ 2 C_{r-1}^1 \int_0^T \int_\Omega \tilde{\sigma}^{r-1} \eta^{p,\gamma} (A_k^l A_p^\gamma - A_k^\gamma A_p^l) \tilde{\sigma}^3 \eta^{k,j} \tilde{\sigma}^{r-2} \tilde{v}^{i,j} \rho_0^2 J^{-1} A_i^l dxdt
\]
\[
+ 2 C_{r-1}^1 \sum_{m=0}^{r-3} C_{r-2}^m \int_0^T \int_\Omega \tilde{\sigma}^{m+1} \eta^{p,\beta} \tilde{\sigma}^{r-2-m} \left[ J(A_k^l A_p^\beta - A_k^\beta A_p^l) \right] \tilde{\sigma}^3 \eta^{k,j} \tilde{\sigma}^{r-2} \tilde{v}^{i,j} \rho_0^2 J^{-2} A_i^l dxdt
\]
\[
+ 2 C_{r-1}^1 \sum_{m=0}^{r-3} C_{r-2}^m \int_0^T \int_\Omega \tilde{\sigma}^{m+1} \eta^{p,\gamma} \tilde{\sigma}^{r-2-m} \left[ J(A_k^l A_p^\gamma - A_k^\gamma A_p^l) \right] \tilde{\sigma}^3 \eta^{k,j} \tilde{\sigma}^{r-2} \tilde{v}^{i,j} \rho_0^2 J^{-2} A_i^l dxdt.
\]
(6.81)
(6.82)
(6.83)
(6.84)
By using the $L^6-L^2-L^3$ Hölder inequality for (6.81) and $L^2-L^6-L^3$ Hölder inequality for (6.82) on higher order terms, together with (6.16) and the Sobolev embedding theorem, we get

$$
| (6.81) | + | (6.82) | \leq M_0 + CTP(\sup_{[0,T]} E_r(t)).
$$

From the $L^m-L^6-L^2-L^3$ Hölder inequality for the case $m = 0$ and $L^6-L^\infty-L^2-L^3$ Hölder inequality for the case $1 \leq m \leq r - 3$ on higher order terms, together with (6.16) and the Sobolev embedding theorem, we can get the same bounds for (6.83) and (6.84). Then, we obtain

$$
| (6.77) | \leq M_0 + CTP(\sup_{[0,T]} E_r(t)).
$$

**Case 3: $s = 2$.** By integration by parts, it yields

$$
-2C_{r-1}^2 \int_0^T \int_\Omega \bar{\sigma}^{-2} d_k \bar{\sigma}^{-3} \eta^k \bar{\sigma}^{r-1} v^j \rho_0^2 J^{-2} A^j_l dxdt
$$

$$
= 2C_{r-1}^2 \int_0^T \int_\Omega \bar{\sigma}^{-1} d_k \bar{\sigma}^{-3} \eta^k \bar{\sigma}^{r-2} v^j \rho_0^2 J^{-2} A^j_l dxdt
$$

$$
+ 2C_{r-1}^2 \int_0^T \int_\Omega \bar{\sigma}^{-2} d_k \bar{\sigma}^4 \eta^k \bar{\sigma}^{r-2} v^j \rho_0^2 J^{-2} A^j_l dxdt
$$

$$
+ 2C_{r-1}^2 \int_0^T \int_\Omega \bar{\sigma}^{-2} d_k \bar{\sigma}^3 \eta^k \bar{\sigma}^{r-2} v^j \rho_0^2 J^{-2} A^j_l dxdt. \tag{6.87}
$$

It is clear that $C_{r-1}^1 (6.86) = C_{r-1}^2 (6.79)$ while the latter has been just estimated. By an $L^6-L^2-L^3$ Hölder inequality for higher order terms, (6.16) and the Sobolev embedding theorem, we get

$$
| (6.87) | + | (6.88) | \leq M_0 + CTP(\sup_{[0,T]} E_r(t)).
$$

That is, (6.85) has the same bound $M_0 + CTP(\sup_{[0,T]} E_r(t))$.

**Case 4: $s = r - 2$ with $r = 5$.** It is easy to see that

$$
\int_0^T \int_\Omega \bar{\sigma}^2 d_k \bar{\sigma}^{-4} \eta^k \bar{\sigma}^4 v^j \rho_0^2 J^{-2} A^j_l dxdt \tag{6.89}
$$

can be bounded by the desired bound in view of Hölder’s inequality and the fundamental theorem of calculus.

Next, we consider (6.75). Since for the case $s = 0$

$$
-2 \int_0^T \int_\Omega \bar{\sigma}^{-1} d_k \bar{\sigma}^3 \eta^k \bar{\sigma}^{r-1} v^j \rho_0^2 J^{-2} A^j_l dxdt = \frac{1}{r-1} (6.77),
$$

and for the case $s = 1$

$$
-2C_{r-1}^1 \int_0^T \int_\Omega \bar{\sigma}^{-2} d_k \bar{\sigma}^3 \eta^k \bar{\sigma}^{r-1} v^j \rho_0^2 J^{-2} A^j_l dxdt = \frac{C_{r-1}^1}{C_{r-1}^2} (6.85),
$$

we have the desired bounds. For the case $s = 2$, we have, by Hölder’s inequality and the Sobolev embedding theorem and the fundamental theorem of calculus, that

$$
\left| -2C_{r-1}^2 \int_0^T \int_\Omega \bar{\sigma}^{-3} d_k \bar{\sigma}^4 \eta^k \bar{\sigma}^{r-1} v^j \rho_0^2 J^{-2} A^j_l dxdt \right|

\leq CTP \sup_{[0,T]} \| \eta \| P(\| \eta \|_3) \| \rho_0 \bar{\sigma}^4 \nabla \eta \|_0 \left( \| \rho_0 \int_0^T \bar{\sigma}^{r-1} \partial_t^2 \nabla \eta dt \|_0 + \| \rho_0 \bar{\sigma}^{r-1} \nabla v(0) \|_0 \right)

\leq M_0 + CTP(\sup_{[0,T]} E_r(t)).
For the case \( s = r - 2 \) and \( r = 5 \),
\[
\left| -2C_{r-1}^2 \int_0^T \int_\Omega \partial_i^2 J^2 A_i^l \partial^r \eta^l_{i,j} \partial^r \eta^{l-1}_{i,j} \rho_0^2 J^{-2} A_i^l \right| \\
\leq CT \sup_{[0,T]} \| \eta \|_1 \| \rho_0 \|_1 \| \nabla \eta \|_1 \left( \| \rho_0 \int_0^T \partial_i^2 \eta^{l-1} \partial^r \eta dt \| + \| \rho_0 \partial_i^2 \eta^{l-1} \nabla \eta(0) \| \right) \\
\leq M_0 + CTP(\sup_{[0,T]} E_r(t)).
\]

For (6.76), by the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus, we can easily obtain the desired bounds. Thus,
\[
\left| \int_0^T (6.71) \right| \leq M_0 + CTP(\sup_{[0,T]} E_r(t)).
\]

Now, we turn to the estimates of \( \int_0^T (6.69) dt \).

**Case 1: \( s = 0 \).** By (2.15) and integration by parts, we see that
\[
2 \int_0^T \int_\Omega \partial_i^2 J^2 A_i^l \partial^r \eta^l_{i,j} \partial^r \eta^{l-1}_{i,j} \rho_0^2 dx dt = -6 \int_0^T \int_\Omega \partial_i^2 J^2 \eta^{l-1}_{i,j} \partial^r \eta^l_{i,j} \partial^r \eta^{l-1}_{i,j} \rho_0^2 J^{-2} dx dt \\
-6 \int_0^T \int_\Omega \partial_i^2 J^2 \eta^{l-1}_{i,j} \partial^r \eta^l_{i,j} \partial^r \eta^{l-1}_{i,j} \rho_0^2 J^{-2} dx dt \\
-6 \sum_{m=0}^{r-3} C_{m+2}^m \int_0^T \int_\Omega \partial_i^2 J^2 \eta^{l-1}_{i,j} \partial^r \eta^l_{i,j} \partial^r \eta^{l-1}_{i,j} \rho_0^2 J^{-2} dx dt \\
= 6 \int_0^T \int_\Omega \partial_i^2 J^2 \eta^{l-1}_{i,j} \partial^r \eta^l_{i,j} \partial^r \eta^{l-1}_{i,j} \rho_0^2 J^{-2} dx dt \\
+6 \int_0^T \int_\Omega \partial_i^2 J^2 \eta^{l-1}_{i,j} \partial^r \eta^l_{i,j} \partial^r \eta^{l-1}_{i,j} \rho_0^2 J^{-2} dx dt \\
+6 \sum_{m=0}^{r-3} C_{m+2}^m \int_0^T \int_\Omega \partial_i^2 J^2 \eta^{l-1}_{i,j} \partial^r \eta^l_{i,j} \partial^r \eta^{l-1}_{i,j} \rho_0^2 J^{-2} dx dt
\]

By the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus, we can easily obtain the desired bound
\[
M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)),
\]
for (6.91) and (6.93)-(6.97).
For (6.92), we use integration by parts with respect to time to get
\[
(6.92) = 6 \int_0^T \nabla^{-1} \eta_{i,k} \nabla \eta_{p,q} \nabla^{-1} \eta_{i,j} \rho_0^2 J^{-1} A_l^k A_p^q A_l^i \, dx dt \tag{6.98}
\]
\[
- 6 \int_0^T \nabla^{-1} v_{i,k} \nabla \eta_{p,q} \nabla^{-1} \eta_{i,j} \rho_0^2 J^{-1} A_l^k A_p^q A_l^i \, dx dt, \tag{6.99}
\]
\[
- 6 \int_0^T \nabla^{-1} \eta_{i,k} \nabla v_{p,q} \nabla^{-1} \eta_{i,j} \rho_0^2 J^{-1} A_l^k A_p^q A_l^i \, dx dt, \tag{6.100}
\]
\[
- 6 \int_0^T \nabla^{-1} \eta_{i,k} \nabla \eta_{p,q} \nabla^{-1} \eta_{i,j} \partial_t \left( \rho_0^2 J^{-1} A_l^k A_p^q A_l^i \right) \, dx dt. \tag{6.101}
\]

It is clear that (6.99) = - (6.92). Thus,
\[
(6.92) = \frac{1}{2} \left( (6.98) + (6.100) + (6.101). \right) \tag{6.102}
\]

Moreover, integration by parts yields
\[
(6.100) = 6 \int_0^T \nabla \nabla^{-1} \eta_{i,k} \nabla \eta_{p,q} \nabla^{-1} \eta_{i,j} \rho_0^2 J^{-1} A_l^k A_p^q A_l^i \, dx dt \tag{6.103}
\]
\[
+ 6 \int_0^T \nabla \nabla^{-1} \eta_{i,k} \nabla^3 \eta_{p,q} \nabla^{-1} \eta_{i,j} \rho_0^2 J^{-1} A_l^k A_p^q A_l^i \, dx dt \tag{6.104}
\]
\[
+ 6 \int_0^T \nabla \nabla^{-1} \eta_{i,k} \nabla^2 \eta_{p,q} \nabla^{-1} \eta_{i,j} \partial_t \left( \rho_0^2 J^{-1} A_l^k A_p^q A_l^i \right) \, dx dt \tag{6.105}
\]

Since (6.104) = (6.92), we have from (6.102)
\[
\]

By integration by parts, we get for \( t = T \)
\[
(6.98) = -12 \int_\Omega \nabla \eta_{i,k} \nabla \eta_{p,q} \nabla^{-1} \eta_{i,j} \rho_0^2 J^{-1} A_l^k A_p^q A_l^i \, dx
\]
\[- 6 \int_\Omega \nabla \eta_{i,k} \nabla \eta_{p,q} \nabla^{-1} \eta_{i,j} \partial_t \left( \rho_0^2 J^{-1} A_l^k A_p^q A_l^i \right) \, dx,
\]

which yields the desired bound for (6.98) by using the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus.

Integration by parts implies
\[
(6.101) = 6 \int_0^T \nabla^{-2} \eta_{i,k} \nabla^3 \eta_{p,q} \nabla^{-1} \eta_{i,j} \partial_t \left( \rho_0^2 J^{-1} A_l^k A_p^q A_l^i \right) \, dx dt \tag{6.107}
\]
\[
+ 6 \int_0^T \nabla^{-2} \eta_{i,k} \nabla^2 \eta_{p,q} \nabla^{-1} \eta_{i,j} \partial_t \left( \rho_0^2 J^{-1} A_l^k A_p^q A_l^i \right) \, dx dt \tag{6.108}
\]
\[
+ 6 \int_0^T \nabla^{-2} \eta_{i,k} \nabla \eta_{p,q} \nabla^{-1} \eta_{i,j} \partial_t \left( \rho_0^2 J^{-1} A_l^k A_p^q A_l^i \right) \, dx dt. \tag{6.109}
\]

Due to (6.107) = - (6.101), it follows that
\[
(6.101) = \frac{1}{2} \left( (6.108) + (6.109) \right),
\]

which implies the desired bound for (6.101) by using the Hölder inequality and the Sobolev embedding theorem. It is clear that both (6.103) and (6.101) have the same bound in view of the Hölder inequality and the Sobolev embedding theorem. Thus, we have obtained
\[
| (6.90) | \leq M_0 + \delta \sup_{[0,T]} E_r(t) + CTP \left( \sup_{[0,T]} E_r(t) \right).
\]
Case 2: $s = 1$. From (2.15), it follows that
\begin{equation}
2C^1_{r-1} \int_0^T \int_\Omega \bar{\nabla}^{r-2} J^{-3} \bar{\nabla}^2 J a_i \partial_j \eta^p \rho_0^2 dxdt
\end{equation}
\begin{equation}
= -6C^1_{r-1} \int_0^T \int_\Omega \bar{\nabla}^{r-3} \left(J^{-3} A^k_i \eta^l \right) \bar{\nabla} \left(J A^q_p \bar{\nabla} \eta^p \right) a_i \partial_j \eta^l \rho_0^2 dxdt
\end{equation}
\begin{equation}
= -6C^1_{r-1} \sum_{m=0}^{r-3} C^m_{r-3} \int_0^T \int_\Omega \bar{\nabla}^{r-3-m} \left(J^{-3} A^k_i \right) \bar{\nabla}^{m+1} \eta^l \partial_j \left(J A^q_p \bar{\nabla} \eta^p \right) \partial^l_j \rho_0^2 \rho_0^2 A^1_i dxdt
\end{equation}
\begin{equation}
- 6C^1_{r-1} \sum_{m=0}^{r-3} C^m_{r-3} \int_0^T \int_\Omega \bar{\nabla}^{r-3-m} \left(J^{-3} A^k_i \right) \bar{\nabla}^{m+1} \eta^l \partial^l_j \eta^p \partial^l_j \rho_0^2 \rho_0^2 J^2 A^1_i A^2_i dxdt.
\end{equation}
By using integration by parts with respect to $\bar{\nabla}$ for (6.111) and the cases $m = 0, r - 3$ in (6.112), and integration by parts with respect to time for the case $m = 1$ in (6.112), together with the Hölder inequality, the Sobolev embedding theorem, (6.16) and the fundamental theorem of calculus, we obtain
\begin{equation}
| (6.110) | \leq M_0 + \delta \sup_{[0,T]} E_r(t) + CTP \left( \sup_{[0,T]} E_r(t) \right).
\end{equation}
Case 3: $s = 2$. By (2.15) and integration by parts, it yields
\begin{equation}
2C^2_{r-1} \int_0^T \int_\Omega \bar{\nabla}^{r-3} \bar{\nabla}^3 \partial^1 \partial^2 J \partial^1 \partial^2 v^j \rho_0^2 \rho_0^2 \left( \bar{\nabla}^2 \partial^1 \partial^2 J \eta^p \rho_0^2 \right)
\end{equation}
\begin{equation}
= -6C^2_{r-1} \sum_{n=0}^{r-4} C^m_{r-4} \sum_{m=0}^{r-4} \int_0^T \int_\Omega \bar{\nabla}^{n+1} \eta^l \partial_j \left(\bar{\nabla}^{r-4-n} J \bar{\nabla}^3 \partial \eta^p \right) \partial^l \partial^j \partial \eta^p \partial^l \rho_0^2 J^2 dxdt
\end{equation}
\begin{equation}
- 6C^2_{r-1} \sum_{n=0}^{r-4} C^m_{r-4} \sum_{m=0}^{r-4} \int_0^T \int_\Omega \bar{\nabla}^{n+1} \eta^l \partial_j \left(\bar{\nabla}^{r-4-n} J \bar{\nabla}^3 \partial \eta^p \right) \partial^l \partial^j \partial \eta^p \partial^l \rho_0^2 J^2 dxdt
\end{equation}
\begin{equation}
= 6C^2_{r-1} \sum_{n=0}^{r-4} C^m_{r-4} \sum_{m=0}^{r-4} \int_0^T \int_\Omega \bar{\nabla}^{n+2} \eta^l \partial_j \left(\bar{\nabla}^{r-4-n} J \bar{\nabla}^3 \partial \eta^p \right) \partial^l \partial^j \partial \eta^p \partial^l \rho_0^2 J^2 dxdt
\end{equation}
\begin{equation}
+ 6C^2_{r-1} \sum_{n=0}^{r-4} C^m_{r-4} \sum_{m=0}^{r-4} \int_0^T \int_\Omega \bar{\nabla}^{n+1} \eta^l \partial_j \left(\bar{\nabla}^{r-4-n} J \bar{\nabla}^3 \partial \eta^p \right) \partial^l \partial^j \partial \eta^p \partial^l \rho_0^2 J^2 dxdt
\end{equation}
\begin{equation}
+ 6C^2_{r-1} \sum_{n=0}^{r-4} C^m_{r-4} \sum_{m=0}^{r-4} \int_0^T \int_\Omega \bar{\nabla}^{n+1} \eta^l \partial_j \left(\bar{\nabla}^{r-4-n} J \bar{\nabla}^3 \partial \eta^p \right) \partial^l \partial^j \partial \eta^p \partial^l \rho_0^2 J^2 dxdt
\end{equation}
\begin{equation}
+ 6C^2_{r-1} \sum_{n=0}^{r-4} C^m_{r-4} \sum_{m=0}^{r-4} \int_0^T \int_\Omega \bar{\nabla}^{n+1} \eta^l \partial_j \left(\bar{\nabla}^{r-4-n} J \bar{\nabla}^3 \partial \eta^p \right) \partial^l \partial^j \partial \eta^p \partial^l \rho_0^2 J^2 dxdt
\end{equation}
\begin{equation}
+ 6C^2_{r-1} \sum_{n=0}^{r-4} C^m_{r-4} \sum_{m=0}^{r-4} \int_0^T \int_\Omega \bar{\nabla}^{n+1} \eta^l \partial_j \left(\bar{\nabla}^{r-4-n} J \bar{\nabla}^3 \partial \eta^p \right) \partial^l \partial^j \partial \eta^p \partial^l \rho_0^2 J^2 dxdt
\end{equation}
Due to the case \( n = 0 \) in (6.114) is equal to \( C_{a-1}^2 \), we have obtained the desired bound for this case of (6.114). For (6.115)-(6.120) and the case \( n = r - 4 \) with \( r = 5 \) of (6.114), we can easily use the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus to get the desired bounds. Then, so does (6.113).

**Case 4: \( s = r - 2 \) with \( r = 5 \).** Integration by parts with respect to time yields

\[
\int_0^T \int_\Omega \nabla J^{-3} \nabla^4 J a_i \nabla^5 v_i \rho_0^2 dx dt \]

\[
= \int_\Omega \nabla J^{-3} \nabla^4 J a_i \nabla^5 v_i \rho_0^2 dx \bigg|_0^T - \int_0^T \int_\Omega \nabla_t (\nabla J^{-3} a_i) \nabla^4 J \nabla^5 v_i \rho_0^2 dx dt \]

\[
- \int_0^T \int_\Omega \nabla J^{-3} \nabla_t \nabla^4 J a_i \nabla^5 v_i \rho_0^2 dx dt - \int_0^T \int_\Omega \nabla J^{-3} \nabla^4 J a_i \nabla^5 v_i \rho_0^2 dx dt,
\]

which can be easily controlled by the desired bound in view of the Hölder inequality and the fundamental theorem of calculus.

Therefore, combining with four cases, we obtain

\[
\left| \int_0^T (6.69) dt \right| \leq M_0 + \delta \sup_{[0,T]} E_r(t) + CT P \left( \sup_{[0,T]} E_r(t) \right). \tag{6.121}
\]

**Step 4. Analysis of the remainder \( \mathcal{J}_4 \).**

For \( l = 0 \), we have from the Hölder inequality, the Sobolev embedding theorem and the Cauchy inequality that

\[
\left| \int_0^T \int_\Omega \nabla^r \rho_0 v_i \nabla^r v dx dt \right| \leq C_T \| \rho_0 \|_{L^1(\Omega)}^{1/2} \left\| \nabla \rho_0 \right\|_{[0,T]} \sup_{[0,T]} \| \rho_0^{1/2} \nabla v \|_0 v_t \|_2
\]

\[
\leq \| \rho_0 \|_2^2 \| \rho_0 \|_r + \delta \sup_{[0,T]} \| \rho_0^{1/2} \nabla v \|_0^2 + CT 4 \| v_t \|_2^4
\]

\[
\leq M_0 + \delta \sup_{[0,T]} E_r(t) + CT P \left( \sup_{[0,T]} E_r(t) \right),
\]

where we need the condition \( \rho_0 \in H^{r+1}(\Omega) \) in view of higher-order Hardy’s inequality.

For \( l = 1 \), we have, at a similar way, that

\[
\left| \int_0^T \int_\Omega \nabla^{r-1} \rho_0 \nabla v_i \nabla^{r-1} v dx dt \right| \leq C_T \| \rho_0 \|_{L^1(\Omega)}^{1/2} \left\| \nabla^{r-1} \rho_0 \right\|_{[0,T]} \sup_{[0,T]} \| \rho_0^{1/2} \nabla v \|_0 v_t \|_3
\]

\[
\leq \| \rho_0 \|_2^2 \| \rho_0 \|_r + \delta \sup_{[0,T]} \| \rho_0^{1/2} \nabla v \|_0^2 + CT 4 \| v_t \|_3^4
\]

\[
\leq M_0 + \delta \sup_{[0,T]} E_r(t) + CT P \left( \sup_{[0,T]} E_r(t) \right).
\]

For \( l = 2 \), we get, by the Hölder inequality and the Sobolev embedding theorem,

\[
\left| \int_0^T \int_\Omega \nabla^{r-2} \rho_0 \nabla^2 v_i \nabla^{r-1} v dx dt \right| \leq C_T \| \rho_0 \|_{L^1(\Omega)}^{1/2} \left\| \nabla^{r-2} \rho_0 \right\|_{[0,T]} \sup_{[0,T]} \| \rho_0^{1/2} \nabla v \|_0 \| \nabla^2 v_t \|_1
\]

\[
\leq \| \rho_0 \|_2^2 \| \rho_0 \|_r + \delta \sup_{[0,T]} \| \rho_0^{1/2} \nabla v \|_0^2 + CT 4 \| v_t \|_3^4
\]
$$\leq M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t))$$.

For \( l = 3 \), we obtain, by the Hölder inequality and the Sobolev embedding theorem, that

$$\left| \int_0^T \int_\Omega \partial^{r-3} \rho_0 \partial^3 \nabla \cdot v^j dxdt \right| \leq C T \| \rho_0 \|_{L^\infty(\Omega)}^{1/2} \left\| \partial^{r-3} \frac{\nabla \cdot v^j}{\rho_0} \right\|_{L^2([0,T])} \sup_{[0,T]} \| \rho_0 \|_0^{1/2} \| v^j \|_0 \| \partial^3 \nabla v^j \|_0$$

$$\leq \| \rho_0 \|_0^2 \| \rho_0 \|_4^4 + \delta \sup_{[0,T]} \| \rho_0 \|_0^{1/2} \| v^j \|_0^2 + C T^4 \| v^j \|_4^4$$

$$\leq M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)).$$

For \( l = 4 \) with \( r = 5 \), we have, by the Hölder inequality and the Sobolev embedding theorem, that

$$\left| \int_0^T \int_\Omega \partial^{r-4} \rho_0 \partial^4 \nabla \cdot v^j dxdt \right| \leq C T \| \rho_0 \|_{L^\infty(\Omega)}^{1/2} \left\| \partial^{r-4} \frac{\nabla \cdot v^j}{\rho_0} \right\|_{L^2([0,T])} \sup_{[0,T]} \| \rho_0 \|_0^{1/2} \| v^j \|_0 \| \partial^4 \nabla v^j \|_0$$

$$\leq \| \rho_0 \|_0^2 \| \rho_0 \|_4^4 + \delta \sup_{[0,T]} \| \rho_0 \|_0^{1/2} \| v^j \|_0^2 + C T^4 \| v^j \|_4^4$$

$$\leq M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)).$$

**Step 5. Analysis of the remainders \( J_5 \) and \( J_6 \).** By integration by parts, (2.16), (1.13b) and (1.13c) for \( l = 1, \cdots, r-1 \), we have

$$\int_0^T J_5 dt = \sum_{i=1}^{r-1} C_i \int_0^T \int_\Omega \partial^{r-i} a_i \partial^l (\rho_0^2 J^{-2}) \partial^r v^i_j dxdt.$$  

This can be written as, by integration by parts with respect to time,

$$\sum_{i=1}^{r-1} C_i \int_\Omega \partial^{r-i} a_i \partial^l (\rho_0^2 J^{-2}) \partial^r \eta^i_j dx \bigg|_{t=T}$$

$$- \sum_{i=1}^{r-1} C_i \int_0^T \int_\Omega \partial^{r-i} a_i \partial^l (\rho_0^2 J^{-2}) \partial^r \eta^i_j dxdt$$

$$- \sum_{i=1}^{r-1} C_i \int_0^T \int_\Omega \partial^{r-i} a_i \partial^l (\rho_0^2 J^{-2}) \partial^r \eta^i_j dxdt.$$  

**Case 1: \( l = 1 \).** By using the fundamental theorem of calculus twice and (2.13), we get for (6.122)

$$\int_\Omega \partial^{r-1} a_i(T) \partial^i (\rho_0^2 J^{-2}(T)) \partial^r \eta^i_j(T) dx$$

$$= \int_\Omega \partial^{r-1} a_i(T) \partial^i \rho_0^2 J^{-2}(T) \partial^r \eta^i_j(T) dx$$

$$- 2 \int_\Omega (\partial^{r-1} a_i(T) \partial^i \rho_0^2 J^{-2}(T) A^q_p(T) \partial^q \eta^i_j(T) dx$$

$$= \int_0^T \int_\Omega \partial^{r-1} a_i dt J^{-2}(T) \partial^i \rho_0^2 \partial^r \eta^i_j(T) dx$$

$$- 2 \int_0^T \int_\Omega \partial^{r-1} a_i dt J^{-2}(T) A^q_p(T) \partial^q \eta^i_j(T) dx$$

$$\leq C T \| \rho_0 \|_0 P(\sup_{[0,T]} E_r(t)) \left( 1 + \| \rho_0 \partial^{r-1} \nabla v_1 \|_0 \right) \| \rho_0 \partial^r \nabla \eta \|_0$$

$$\leq M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)).$$
Similarly, we have for (6.123)
\[ \left| \int_0^T \int_\Omega \partial_d^{-1} a_i^i \partial_d (\rho^2 J^{-2}) \partial_d^\tau \eta_{i,j} \, dx \, dt \right| \leq M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)). \]

For (6.124), it is harder to be controlled than (6.122) and (6.123). We write it as
\[ -\int_0^T \int_\Omega \partial_d^{-1} a_i^i \partial_d (\rho^2 J^{-2}) \partial_d^\tau \eta_{i,j} \, dx \, dt \]
\[ = 2 \int_0^T \int_\Omega \partial_d^{-1} a_i^i \rho_0^2 J^{-2} A_q^p v_A \partial_d^\tau \eta_{i,j} \, dx \, dt \]
\[ + 2 \int_0^T \int_\Omega \partial_d^{-1} a_i^i \rho_0^2 J^{-2} A_q^p v_A \partial_d^\tau \eta_{i,j} \, dx \, dt \]
\[ + 2 \int_0^T \int_\Omega \partial_d^{-1} a_i^i \rho_0^2 J^{-2} A_q^p v_A \partial_d^\tau \eta_{i,j} \, dx \, dt \]
\[ + 2 \int_0^T \int_\Omega \partial_d^{-1} a_i^i \rho_0^2 J^{-2} A_q^p v_A \partial_d^\tau \eta_{i,j} \, dx \, dt. \]

It is easy to see that (6.125) and (6.126) are bounded by
\[ M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)). \]

By integration by parts, Hölder’s inequality, (6.16) and Sobolev’s embedding theorem, it holds
\[ (6.127) = -2 \int_0^T \int_\Omega \partial_d^{-1} a_i^i A_q^p v_A \partial_d^\tau \eta_{i,j} \rho_0^2 J^{-2} \, dx \, dt \]
\[ \leq \int_0^T \| \rho_0 \partial^\tau a \|_{L^1} \| \eta \|_{L^2} \| \partial_d v \|_{L^2} \| \rho_0 \partial^\tau \eta \|_{L^1} \, dt \]
\[ + \int_0^T \| \rho_0 \|_{L^1} \| \partial^\tau a \|_{L^1} \| \eta \|_{L^3} \| \partial_d v \|_{L^2} \| \rho_0 \partial^\tau \eta \|_{L^1} \, dt \]
\[ + \int_0^T \| \rho_0 \|_{L^1} \| \partial^\tau a \|_{L^1} \| \eta \|_{L^3} \| \partial_d v \|_{L^2} \| \rho_0 \partial^\tau \eta \|_{L^1} \, dt \]
\[ + \int_0^T \| \rho_0 \|_{L^1} \| \partial^\tau a \|_{L^1} \| \eta \|_{L^3} \| \partial_d v \|_{L^2} \| \rho_0 \partial^\tau \eta \|_{L^1} \, dt \]
\[ \leq M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)). \]

For (6.128), we can easily get the desired bound by an $L^6 - L^3 - L^2$ Hölder’s inequality and the Sobolev embedding theorem because each component of $a_i^i$ is quadratic in $\partial_d \eta$ due to (1.9).
Case 2: \( l = 2 \). By the fundamental theorem of calculus, Hölder’s inequality and the Sobolev embedding theorem, we can see that

\[
\left| \int \Omega \frac{\partial}{\partial t} \left( \rho_0^2 J^{-2} \right) \nabla \eta \cdot d \nabla \eta \right|_{\Omega} \leq M_0 + \delta \sup_{[0,T]} E_r(t) + CT P(\sup_{[0,T]} E_r(t)).
\]

From (2.18), the Hölder inequality, the Sobolev embedding theorem and (6.16), it yields

\[
- \int_0^T \int \Omega \frac{\partial}{\partial t} \left( J^{-2} \rho_0^2 \right) \nabla \eta \cdot d \nabla \eta \, dx \, dt
= \int_0^T \int \Omega \frac{\partial}{\partial t} \left( J^{-2} \rho_0^2 \right) \nabla \eta \cdot d \nabla \eta \, dx \, dt
= -2 \int_0^T \int \Omega J^{-2} \rho_0^2 \left( A_i \partial_i A_j - A_j \partial_i A_i \right) \nabla \eta \cdot d \nabla \eta \, dx \, dt + \text{reminders}
\leq C \left\| \rho_0 \right\|_2 \int_0^T \left\| \nabla \eta \right\|_{L^2(\Omega)} \left\| \nabla \eta \right\|_{L^2(\Omega)} \left\| \rho_0 \nabla \eta \right\|_{L^2(\Omega)} dt + \text{reminders}
\leq M_0 + \delta \sup_{[0,T]} E_r(t) + CT P(\sup_{[0,T]} E_r(t)),
\]

since the remainders can be easily controlled by the desired bound.

Similarly, we can get the bound for the last integral

\[
\left| \int_0^T \int \Omega \frac{\partial}{\partial t} \left( \rho_0^2 \partial_i J^{-2} \right) \nabla \eta \cdot d \nabla \eta \, dx \, dt \right| \leq M_0 + \delta \sup_{[0,T]} E_r(t) + CT P(\sup_{[0,T]} E_r(t)).
\]

Case 3: \( l = 3 \). By using the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus, the spatial integral (6.122) can be bounded by \( M_0 + \delta \sup_{[0,T]} E_r(t) + CT P(\sup_{[0,T]} E_r(t)) \). Similarly, we can get the same bound for the first space-time integral (6.123). Since the norm \( \| \rho_0 \partial_i ^2 J^{-2} \|_3 \) is contained in the energy function \( E_r(t) \), the last space-time integral (6.124) have the same bound by the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus.

Case 4: \( l = r - 1 \) and \( r = 5 \). They can be easily controlled by the desired bound, especially with the help of the fundamental theorem of calculus for (6.124).

We can deal with the integrals in \( \mathcal{J}_6 \) by using a similar argument and we omit the details for simplicity.

Step 6. Summing identities. Integrating (6.1) over \([0,T]\), by Hölder’s inequality and Cauchy’s inequality, we have, for sufficiently small \( T \) such that

\[
\frac{1}{2} \int_0^T \left\| \varphi^I \right\|^2 dx + \frac{1}{2} \int_0^T \left[ \left\| \nabla \varphi^I \right\|^2 + \left| \text{div} \varphi^I \right|^2 - \left| \text{curl} \varphi^I \right|^2 \right] \rho_0^2 J^{-1} \, dx
\geq \frac{1}{2} \int_0^T \left\| \varphi^I_0 \right\|^2 dx - \frac{1}{2} \int_0^T \left[ \left\| \nabla \varphi^I \right\|^2 + \left| \text{div} \varphi^I \right|^2 - \left| \text{curl} \varphi^I \right|^2 \right] \rho_0^2 J^{-1} \text{div}_\eta \, dx
+ \frac{1}{2} \int_0^T \int \Omega \nabla \eta \cdot \nabla \varphi^I_0 \, dx \, dt - \int_0^T \int_0^T \nabla \eta \cdot (A_i \partial_i \varphi^I J^{-1}) \, dx \, dt - \int_0^T \int_0^T \nabla \eta \cdot (A_i \partial_i \varphi^I J^{-1}) \, dx \, dt
+ \int_0^T \int_0^T \nabla \eta \cdot (T \varphi^I) J^{-1} \, dx \, dt - \int_0^T \left[ (6.4) + (6.73) + (6.71) + (6.69) \right] \, dt + \int_0^T [\mathcal{J}_4 + \mathcal{J}_5 + \mathcal{J}_6] \, dt
\leq M_0 + \delta \sup_{[0,T]} E_r(t) + CT P(\sup_{[0,T]} E_r(t)).
\]

By the fundamental theorem of calculus, we have

\[
\int \left| \nabla \varphi^I \right|^2 \rho_0^2 J^{-1} \, dx = \int \left( \nabla \eta \varphi^I \right) \left( \nabla \eta \varphi^I \right) \rho_0^2 J^{-1} \, dx = \int \nabla \varphi^I \cdot A_i \nabla \eta \rho_0^2 \, dx
\]
\[= \int_{\Omega} \rho_0^2 \left( \int_0^t (A_j^{-1})_b(t')dt' + A_j(0) \right) \nabla \eta_{i,s} \left( \int_0^t A^0_{i}(t')dt' + A^0(0) \right) \nabla \eta_{j,p} \, dx\]

\[= \int_{\Omega} \rho_0^2 |\nabla \eta_{i,s}|^2 \, dx + \int_{\Omega} \rho_0^2 \left( \int_0^t \nabla (A_j^{-1})_b(t') \nabla \eta_{i,s} \left( \int_0^t A^0_{i}(t')dt' + A^0(0) \right) \nabla \eta_{j,p} \, dx\right) + \int_{\Omega} \rho_0^2 \nabla (A_j^{-1})_b(t') \nabla \eta_{i,s} \left( \int_0^t A^0_{i}(t')dt' \right) \nabla \eta_{j,p} \, dx,\]

which yields

\[\sup_{[0,T]} \left| \rho_0 \nabla \eta \nabla \eta \right|^2_0 - \left| \rho_0 \nabla \nabla \eta \right|^2_0 \leq CT^2 \sup_{[0,T]} \left| \rho_0 \nabla \eta \eta \right|^2_0 \|v\|^2_{r-1} \|\eta\|^8 + CT \sup_{[0,T]} \left| \rho_0 \nabla \nabla \eta \eta \right|^2_0 \|v\|^r_{r-1} \|\eta\|^4.\]

Thus, taking \(T\) so small that \(CT \|v\|_{r-1} \|\eta\|^4 \leq 1/6\), we obtain

\[\frac{1}{2} \sup_{[0,T]} \left| \rho_0 \nabla \eta \nabla \eta \right|_0 \leq \sup_{[0,T]} \left| \rho_0 \nabla \eta \nabla \eta \right|_0 \leq \frac{3}{2} \sup_{[0,T]} \left| \rho_0 \nabla \eta \eta \right|_0. \quad (6.129)\]

Similarly, we have

\[\int_{\Omega} |\nabla \eta_{i,s}|^2 \rho_0^2 J^{-1} \, dx\]

\[= \int_{\Omega} \rho_0^2 \left( \nabla \eta_{i,s} + \epsilon_{ij} \nabla \eta_{i,s} \int_0^t \nabla (A_j^{-1})_b(t') \nabla \eta_{i,s} \left( \int_0^t A^0_{i}(t')dt' \right) \nabla \eta_{j,p} \, dx\right) + \int_{\Omega} \rho_0^2 \nabla \eta_{i,s} \left( \int_0^t A^0_{i}(t')dt' \right) \nabla \eta_{j,p} \, dx,\]

which yields for a sufficiently small \(T\) that

\[\frac{1}{2} \sup_{[0,T]} \left| \rho_0 \nabla \eta \nabla \eta \right|_0 \leq \sup_{[0,T]} \left| \rho_0 \nabla \eta \nabla \eta \right|_0 \leq \frac{3}{2} \sup_{[0,T]} \left| \rho_0 \nabla \eta \nabla \eta \right|_0, \quad (6.132)\]

\[\frac{1}{2} \sup_{[0,T]} \left| \rho_0 \nabla \eta \nabla \eta \right|_0 \leq \sup_{[0,T]} \left| \rho_0 \nabla \eta \nabla \eta \right|_0 \leq \frac{3}{2} \sup_{[0,T]} \left| \rho_0 \nabla \eta \nabla \eta \right|_0. \quad (6.133)\]

Thus, we obtain the desired inner estimates with the help of curl estimates.
Step 7: Boundary estimates. By Lemma 2.4, (2.5), the fundamental theorem of calculus and Hölder’s inequality, we obtain
\[ |\partial^r \eta|^2_{2/1} \lesssim |\partial^r \eta|_0^2 + \|\text{curl} \partial^{r-1} \eta\|_0^2 \]
\[ \lesssim \|\rho_0 \partial^r \eta\|_0^2 + \|\rho_0 \nabla \partial^r \eta\|_0^2 + \|\text{curl} \partial^{r-1} \eta\|_0^2 \]
\[ \lesssim \|\rho_0\int_0^t \partial^r v dt\|_0^2 + \|\rho_0 \nabla \partial^r \eta\|_0^2 + \|\text{curl} \partial^{r-1} \eta\|_0^2 \]
\[ \lesssim \|\rho_0\|_{L^2(0,T)} \left[ \|\rho_0^{1/2} \partial^r v\|_0^2 + \|\rho_0 \partial^r \nabla \eta\|_0^2 + \|\text{curl} \partial^{r-1} \eta\|_0^2 \right], \]
which implies the desired estimates from curl estimates. \qed

7. The estimates for the time derivatives

We have the following estimates.

Proposition 7.1. Let \( r \in \{4, 5\} \). Then, for a small \( \delta > 0 \) and the constant \( M_0 \) depend on \( 1/\delta \), we have
\[ \sup_{[0,T]} \left[ \|\rho_0^{1/2} \partial^r v\|_0^2 + \|\rho_0 \partial^r \nabla \eta\|_0^2 + \|\rho_0 \partial^r \text{div} \eta\|_0^2 \right] \leq M_0 + \delta \sup_{[0,T]} E_5(t) + CT \rho_0 \sup_{[0,T]} E_5(t). \]

Proof. Acting \( \partial^r \) on (1.13a), and taking the \( L^2(\Omega) \)-inner product with \( \partial^r v^i \), we obtain
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega \rho_0 |\partial^r v|^2 dx + \mathcal{J}_1 + \mathcal{J}_2 = \mathcal{J}_3, \]
where
\[ \mathcal{J}_1 := \int_\Omega \partial^r a_j^i (\rho_0^2 J^{-2})_{,j} \partial^r v^i dx, \quad \mathcal{J}_2 := \int_\Omega a_j^i (\rho_0^2 \partial^r J^{-2})_{,j} \partial^r v^i dx, \]
\[ \mathcal{J}_3 := -\sum_{r=1}^{2r-1} C_{2r}^r \int_\Omega \partial^r a_j^i (\rho_0^2 J^{-2})_{,j} \partial^r v^i dx. \]

Step 1. Analysis of the integral \( \mathcal{J}_1 \). Integration by parts gives, by noticing that \( \partial^r \partial^3 J = 0 \) on \( \Gamma_0 \), that
\[ \mathcal{J}_1 = -\int_\Omega \partial^2 \partial^3 a_j^i \partial^2 v^i,_{,j} \rho_0^2 J^{-2} dx + \int_{\Gamma_0} \partial^2 \partial^3 a_j^i \partial^2 v^i, \rho_0^2 J^{-2} d\Omega_{12} \]
\[ = -\int_\Omega \partial^{r-1} (v^i,_{,k} J^{-1} (a_j^i a_r^j - a_r^i a_j^j)) \partial^r v^i,_{,j} \rho_0^2 J^{-2} dx \]
\[ = -\int_\Omega \partial^{r-1} v^i,_{,k} \partial^2 v^i, a^k, a_j^j J^{-1} dx \]
\[ + \int_\Omega \partial^{r-1} v^i,_{,k} \partial^2 v^i, a^k, a_j^j \rho_0^2 J^{-1} dx \]
\[ - \sum_{s=1}^{2r-1} C_{2r}^s \int_\Omega \partial^{r-1-s} v^i,_{,k} \partial^r (J^{-1} (a^i a^k - a^k a^i)) \partial^2 v^i,_{,j} \rho_0^2 J^{-2} dx. \]

Then, we have
\[ (7.1) = -\frac{1}{2} \frac{d}{dt} \int_\Omega |\text{div} \partial^r v|^2 \rho_0^2 J^{-1} dx + \frac{1}{2} \int_\Omega \partial^r v^i,_{,k} \partial^r v^i,_{,j} \partial^s (a^k a^j) \rho_0^2 J^{-1} dx \]
\[ + \frac{1}{2} \int_\Omega |\text{div} \partial^r v|^2 \rho_0^2 \partial^s J^{-1} dx. \]
It follows from (2.2) that
\[
\frac{d}{dt} \partial_t^{2r-1} v_j A^k_i \partial_t^{2r-1} v_i = \frac{1}{2} \partial_t \left[ |\nabla \eta \partial_t^{2r-1} v|^2 - |\text{curl} \eta \partial_t^{2r-1} v|^2 \right] - \frac{1}{2} \partial_t^{2r-1} v_j \partial_t^{2r-1} v_i \partial_t (A^k_i A_j^k).
\]
Thus, we have
\[
(7.2) = \frac{1}{2} \frac{d}{dt} \int_\Omega \left[ |\nabla \eta \partial_t^{2r-1} v|^2 - |\text{curl} \eta \partial_t^{2r-1} v|^2 \right] \rho_0^2 J^{-1} \, dx \\
- \frac{1}{2} \int_\Omega \left[ |\nabla \eta \partial_t^{2r-1} v|^2 - |\text{curl} \eta \partial_t^{2r-1} v|^2 \right] \rho_0^2 \partial_t J^{-1} \, dx \\
- \frac{1}{2} \int_\Omega \partial_t^{2r-1} v_j \partial_t^{2r-1} v_i \partial_t (A^k_i A_j^k) \rho_0^2 J^{-1} \, dx.
\]

**Step 2. Analysis of the integral \( J_2 \).** Similar to those of \( J_1 \), by noticing that \( a^3 \partial_t^{2r} v^j = 0 \) on \( \Gamma_0 \), we get
\[
J_2 = - \int_\Omega a^j_i \partial_t^{2r} v^j \, dx + \int_\Omega a^3_i \partial_t^{2r} v^j \, dx + \int_\Omega a^3_i \partial_t^{2r} v^j \, dx
\]
\[
= 2 \int_\Omega a^j_i \partial_t^{2r} v^j \, dx + \int_\Omega a^3_i \partial_t^{2r} v^j \, dx
\]
\[
+ 2 \sum_{s=0}^{2r-2} C_{2r-1}^s \int_\Omega a^j_i \partial_t^{2r} v^j \, dx
\]
\[
= - 2 \cdot (7.1) + (7.4).
\]

Thus, we obtain by integrating over \([0, T] \]
\[
\frac{1}{2} \int_\Omega \rho_0 |\partial_t^{2r} v|^2 \, dx + \frac{1}{2} \int_\Omega \left[ |\nabla \eta \partial_t^{2r-1} v|^2 + |\text{div} \eta \partial_t^{2r-1} v|^2 - |\text{curl} \eta \partial_t^{2r-1} v|^2 \right] \rho_0^2 J^{-1} \, dx
\]
\[
= 1 \int_\Omega \rho_0 |\partial_t^{2r} v(0)|^2 \, dx + \frac{1}{2} \int_\Omega \left[ |\nabla \partial_t^{2r-1} v(0)|^2 + |\text{div} \partial_t^{2r-1} v(0)|^2 - |\text{curl} \partial_t^{2r-1} v(0)|^2 \right] \rho_0^2 \, dx
\]
\[
+ \frac{1}{2} \int_0^T \int_\Omega \left[ |\nabla \eta \partial_t^{2r-1} v|^2 + |\text{div} \eta \partial_t^{2r-1} v|^2 - |\text{curl} \eta \partial_t^{2r-1} v|^2 \right] \rho_0^2 \partial_t J^{-1} \, dx dt
\]
\[
+ \frac{1}{2} \int_0^T \int_\Omega \partial_t^{2r-1} v_j \partial_t^{2r-1} v_i \partial_t (A^k_i A_j^k) \rho_0^2 J^{-1} \, dx dt
\]
\[
+ \frac{1}{2} \int_0^T \int_\Omega \partial_t^{2r-1} v_j \partial_t^{2r-1} v_i \partial_t (A^k_i A_j^k) \rho_0^2 J^{-1} \, dx dt
\]
\[
- \int_0^T (7.3) \, dt - \int_0^T (7.4) \, dt + \int_0^T J_3 \, dt + \int_0^T J_4 \, dt.
\]

The first three space-time double integrals can be absorbed by the left hand side as long as \( T \) is sufficiently small.

**Step 3. Analysis of the remainder \( \int_0^T (7.3) \, dt \).** Integration by parts with respect to time gives
\[
- \int_0^T (7.3) \, dt = \sum_{s=1}^{2r-1} C_{2r-1}^s \int_0^T \int_\Omega \partial_t^{2r-1-s} v_j \partial_t^s (J(A^k_i A_j^k - A^k_i A_j^k)) \partial_t^{2r} v_i \, dx dt
\]
\[
= \sum_{s=1}^{2r-1} C_{2r-1}^s \int_\Omega \partial_t^{2r-s} \eta_j \partial_t^s (J(A^k_i A_j^k - A^k_i A_j^k)) \partial_t^{2r} \eta_i \rho_0^2 J^{-2} \, dx \bigg|_0^T
\]
\[
- \sum_{s=1}^{2r-1} C_{2r-1}^s \int_\Omega \partial_t^{2r+1-s} \eta_j \partial_t^s (J(A^k_i A_j^k - A^k_i A_j^k)) \partial_t^{2r} \eta_i \rho_0^2 J^{-2} \, dx dt
\]
We first consider (7.6). For the cases $s = 1, 2$, it is easy to get the desired bounds by the Hölder inequality and the Sobolev embedding theorem. For the case $s = 3$, we have
\begin{equation}
\int_{0}^{T} \int_{\Omega} \partial_{t}^{2r-2} \eta_{t,k} \partial_{t}^{3} (J(A^{j}_{i}A^{k}_{j} - A^{k}_{j}A^{j}_{i})) \partial_{t}^{2r} \eta_{ij} \rho_{ij}^{2} J^{-2} dx dt
\end{equation}
(7.7)
\begin{equation}
\int_{0}^{T} \int_{\Omega} \partial_{t}^{2r-2} \eta_{t,k} \partial_{t}^{3} (J(A^{j}_{i}A^{k}_{j} - A^{k}_{j}A^{j}_{i})) \partial_{t}^{2r} \eta_{ij} \rho_{ij}^{2} \partial_{t}^{2} J^{-2} dx dt.
\end{equation}
(7.8)
It is clear that (7.11) is easy to deal with by an $L^{6} - L^{3} - L^{2}$ Hölder inequality. For (7.10), integration by parts yields
\begin{equation}
\int_{0}^{T} \int_{\Omega} \partial_{t}^{2r-2} \eta_{t,k} \partial_{t}^{3} (J(A^{j}_{i}A^{k}_{j} - A^{k}_{j}A^{j}_{i})) \partial_{t}^{2r} \eta_{ij} \rho_{ij}^{2} J^{-2} dx dt
\end{equation}
(7.12)
\begin{equation}
\int_{0}^{T} \int_{\Omega} \partial_{t}^{2r-2} \eta_{t,k} \partial_{t}^{3} (J(A^{j}_{i}A^{k}_{j} - A^{k}_{j}A^{j}_{i})) \partial_{t}^{2r} \eta_{ij} \rho_{ij}^{2} \partial_{t}^{2} J^{-2} dx dt
\end{equation}
(7.13)
\begin{equation}
\int_{0}^{T} \int_{\Omega} \partial_{t}^{2r-2} \eta_{t,k} \partial_{t}^{3} (J(A^{j}_{i}A^{k}_{j} - A^{k}_{j}A^{j}_{i})) \partial_{t}^{2r} \eta_{ij} \rho_{ij}^{2} J^{-2} dx dt.
\end{equation}
(7.14)
which, then, can be controlled easily by the desired bound by an $L^{6} - L^{3} - L^{5}$ Hölder inequality, in addition for (7.13), with the help of
\begin{equation}
\left\| \partial_{t}^{2m} D^{-1-m} \eta \right\|_{L^{2}([0,T] \times \Omega)}^{2} \leq CT^{2/3} \left[ \left\| \partial_{t}^{2m} D^{-1-m} v(0) \right\|_{L^{2}(\Omega)}^{2} + \sup_{[0,T]} \left\| \partial_{t}^{2m} D^{-1-m} \eta \right\|_{L^{2}(\Omega)} \right]
\end{equation}
(7.15)
for the integer $0 \leq m \leq r - 1$ due to (3.1). For the case $s = 4$, we have
\begin{equation}
\int_{0}^{T} \int_{\Omega} \partial_{t}^{2r-3} \eta_{t,k} \partial_{t}^{4} (J(A^{j}_{i}A^{k}_{j} - A^{k}_{j}A^{j}_{i})) \partial_{t}^{2r} \eta_{ij} \rho_{ij}^{2} J^{-2} dx dt
\end{equation}
(7.16)
\begin{equation}
= \int_{0}^{T} \int_{\Omega} \partial_{t}^{2r-3} \eta_{t,k} \partial_{t}^{4} (J(A^{j}_{i}A^{k}_{j} - A^{k}_{j}A^{j}_{i})) \partial_{t}^{2r} \eta_{ij} \rho_{ij}^{2} J^{-2} dx dt
\end{equation}
(7.17)
\begin{equation}
\int_{0}^{T} \int_{\Omega} \partial_{t}^{2r-3} \eta_{t,k} \partial_{t}^{4} (J(A^{j}_{i}A^{k}_{j} - A^{k}_{j}A^{j}_{i})) \partial_{t}^{2r} \eta_{ij} \rho_{ij}^{2} J^{-2} dx dt.
\end{equation}
(7.18)
It is clear that (7.18) is easy to deal with by an $L^{6} - L^{3} - L^{2}$ Hölder inequality. For (7.17), integration by parts yields
\begin{equation}
\int_{0}^{T} \int_{\Omega} \partial_{t}^{2r-3} \eta_{t,k} \partial_{t}^{4} (J(A^{j}_{i}A^{k}_{j} - A^{k}_{j}A^{j}_{i})) \partial_{t}^{2r} \eta_{ij} \rho_{ij}^{2} J^{-2} dx dt
\end{equation}
(7.19)
\begin{equation}
\int_{0}^{T} \int_{\Omega} \partial_{t}^{2r-3} \eta_{t,k} \partial_{t}^{4} (J(A^{j}_{i}A^{k}_{j} - A^{k}_{j}A^{j}_{i})) \partial_{t}^{2r} \eta_{ij} \rho_{ij}^{2} J^{-2} dx dt
\end{equation}
(7.20)
By an $L^2$-$L^3$-$L^6$ Hölder inequality, (7.19) and (7.21) can be easily estimated. We can also use an $L^3$-$L^2$-$L^6$ Hölder inequality to control (7.20) since $\partial_t^2 \nabla \eta \in L^3([0,T] \times \Omega)$ due to (7.15). For the case $s = 5$,

$$\int_0^T \int_\Omega \partial_t^{2r-3} \eta^i \partial_t^{2r-3} \eta^j \partial_t^5 \left( J(A^\beta_i A^k_i - A^\beta_i A^m_i) \right) \partial_t^2 \eta^i \rho_0^2 J^{-2} dx dt.$$  

(7.21)

For (7.30), it can be controlled by $E_4(t)$ instead of $E_5(t)$ when $r = 4$; while we can get the desired bound $M_0 + \delta \sup_{[0,T]} E_5(t) + C \sup_{[0,T]} E_5(t)$ for $r = 5$ by an $L^5$-$L^2$-$L^6$ Hölder inequality. For (7.29), integration by parts implies

$$\int_0^T \int_\Omega \partial_t^{2r-5} \eta^i \partial_t^{2r-5} \eta^j \partial_t^6 \left( J(A^\beta_i A^k_i - A^\beta_i A^m_i) \right) \partial_t^2 \eta^j \rho_0^2 J^{-2} dx dt.$$  

(7.28)

By an $L^2$-$L^3$-$L^6$ Hölder inequality, (7.25) and (7.27) can be easily estimated. We can use an $L^3$-$L^2$-$L^6$ Hölder inequality to control (7.20) because of $\rho_0 \partial_t^5 \nabla \eta \in L^2(\Omega)$ in view of the fundamental theorem of calculus.

For the case $s = 6$,

$$\int_0^T \int_\Omega \partial_t^{2r-5} \eta^i \partial_t^{2r-5} \eta^j \partial_t^6 \left( J(A^\beta_i A^k_i - A^\beta_i A^m_i) \right) \partial_t^2 \eta^j \rho_0^2 J^{-2} dx dt.$$  

(7.29)

For (7.30), it can be controlled by $E_5(t)$ instead of $E_4(t)$ when $r = 4$; while we can get the desired bound $M_0 + \delta \sup_{[0,T]} E_5(t) + C \sup_{[0,T]} E_5(t)$ for $r = 5$ by an $L^5$-$L^2$-$L^6$ Hölder inequality. For (7.29), integration by parts implies

$$\int_0^T \int_\Omega \partial_t^{2r-5} \eta^i \partial_t^{2r-5} \eta^j \partial_t^6 \left( J(A^\beta_i A^k_i - A^\beta_i A^m_i) \right) \partial_t^2 \eta^j \rho_0^2 J^{-2} dx dt.$$  

(7.29)

which can be controlled by the desired bound in view of the $L^3$-$L^2$-$L^6$ Hölder inequality and with the help of (7.15) additionally for (7.31).

For the cases $s = 7, 8, 9$ with $r = 5$, they are easy to be controlled by the desired bound via the Hölder inequality.

Next, we consider (7.7). For the cases $s = 1, 6, 7$, and $s = 8, 9$ with $r = 5$, it is easy to get the desired bounds by the Hölder inequality and the Sobolev embedding theorem. For the case
By integration by parts, we have
\[
\int_0^T \int_\Omega \partial_t \eta_i \cdot \partial_j \left( J(A_i^j A_j^i - A_i^j A_j^i) \right) \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt
= \int_0^T \int_\Omega \partial_t \eta_i \cdot \partial_j \left( J(A_i^j A_j^i - A_i^j A_j^i) \right) \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt
+ \int_0^T \int_\Omega \partial_t \eta_i \cdot \partial_j \left( J(A_i^j A_j^i - A_i^j A_j^i) \right) \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt. \tag{7.34}
\]

By integration by parts, we have
\[
\tag{7.34} = - \int_0^T \int_\Omega \partial_t \eta_i \cdot \partial_j \left( J(A_i^j A_j^i - A_i^j A_j^i) \right) \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt \tag{7.36}
- \int_0^T \int_\Omega \partial_t \eta_i \cdot \partial_j \left( J(A_i^j A_j^i - A_i^j A_j^i) \right) \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt \tag{7.37}
- \int_0^T \int_\Omega \partial_t \eta_i \cdot \partial_j \left( J(A_i^j A_j^i - A_i^j A_j^i) \right) \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt. \tag{7.38}
\]

The $L^2$-$L^3$-$L^6$ Hölder inequality gives the desired bounds for (7.36) and (7.38). For (7.37), we can use an $L^2$-$L^3$-$L^6$ Hölder inequality and (7.15) to get the bound. For (7.35), we can obtain the desired bound by using an $L^6$-$L^3$-$L^2$ Hölder inequality.

For the case $s = 3$, by an $L^3$-$L^6$-$L^2$ Hölder inequality and (7.15), we get
\[
\left| \int_0^T \int_\Omega \partial_t \eta_i \cdot \partial_j \left( J(A_i^j A_j^i - A_i^j A_j^i) \right) \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt \right|
\leq C T^{2/3} \| \rho_0 \|_2 \| \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt \sup_{[0, T]} \| \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt \tag{7.39}
\leq M_0 + \delta \sup_{[0, T]} \| \rho_0 \|_2 \| \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt \tag{7.40}
\]

For the case $s = 4$, by an $L^6$-$L^3$-$L^2$ Hölder inequality and (7.15), we get
\[
\left| \int_0^T \int_\Omega \partial_t \eta_i \cdot \partial_j \left( J(A_i^j A_j^i - A_i^j A_j^i) \right) \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt \right|
\leq C T^{2/3} \| \rho_0 \|_2 \| \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt \sup_{[0, T]} \| \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt \tag{7.39}
\leq M_0 + \delta \sup_{[0, T]} \| \rho_0 \|_2 \| \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt \tag{7.40}
\]

For (7.39), integration by parts gives
\[
\tag{7.39} = - \int_0^T \int_\Omega \partial_t \eta_i \cdot \partial_j \left( J(A_i^j A_j^i - A_i^j A_j^i) \right) \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt
- \int_0^T \int_\Omega \partial_t \eta_i \cdot \partial_j \left( J(A_i^j A_j^i - A_i^j A_j^i) \right) \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt
- \int_0^T \int_\Omega \partial_t \eta_i \cdot \partial_j \left( J(A_i^j A_j^i - A_i^j A_j^i) \right) \partial_t \eta_i \cdot \rho_0^2 J^{-2} \ dx \ dt.
\]
\[- \int_0^T \int_\Omega \partial_t^{2r-5} \eta^i j \partial^6 (J(A_t^k A_t^k - A_t^k A_t^k)) \partial_t^{2r} \eta^i (\rho_0^2 J^{-2}, \beta) dx dt,\]

which has the desired bound by using an $L^3 - L^2 - L^6$ Hölder inequality for three double integrals and (7.15) for the first integral additionally. For the integral (7.40), we have to use $E_5(t)$ to control it, especially for $r = 4$, as the same as the case $s = 6$ in (7.6).

The spatial integral (7.5) can be treated similarly as for (7.7) with the help of the fundamental theorem of calculus for one lower order term in order to get the factor $T$. For the last integral (7.8), it is much easier to get the bound than (7.7), thus we omit the details. Therefore, we have obtained

\[ \left| \int_0^T (7.3) dt \right| \leq M_0 + \delta \sup_{[0,T]} E_5(t) + CTP\left( \sup_{[0,T]} E_5(t) \right). \]

**Step 4. Analysis of the remainder $\int_0^T (7.4) dt$.** By integration by parts with respect to time, we get

\[
\int_0^T (7.4) dt = -2 \sum_{s=0}^{2r-2} \sum_{i,j} C_{2s}^r \int_0^T \int_\Omega \partial_t^{2r} \eta^i j \partial_t^{2r-1-s} (J^{-2} A_t^k) \partial_t^s \eta^i j \partial_t \rho_0^2 J^{-2} A_t^k \partial_t dx dt

= -2 \sum_{s=0}^{2r-2} \sum_{i,j} C_{2s}^r \int_0^T \int_\Omega \partial_t^{2r} \eta^i j \partial_t^{2r-1-s} (J^{-2} A_t^k) \partial_t^s \eta^i j \partial_t \rho_0^2 J^{-2} A_t^k \partial_t dx dt (7.41)

\[-2 \sum_{s=0}^{2r-2} \sum_{i,j} C_{2s}^r \int_0^T \int_\Omega \partial_t^{2r} \eta^i j \partial_t^{2r-1-s} (J^{-2} A_t^k) \partial_t^s \eta^i j \partial_t \rho_0^2 J^{-2} A_t^k \partial_t dx dt (7.42)

\[-2 \sum_{s=0}^{2r-2} \sum_{i,j} C_{2s}^r \int_0^T \int_\Omega \partial_t^{2r} \eta^i j \partial_t^{2r-1-s} (J^{-2} A_t^k) \partial_t^s \eta^i j \partial_t \rho_0^2 J^{-2} A_t^k \partial_t dx dt (7.43)

\[-2 \sum_{s=0}^{2r-2} \sum_{i,j} C_{2s}^r \int_0^T \int_\Omega \partial_t^{2r} \eta^i j \partial_t^{2r-1-s} (J^{-2} A_t^k) \partial_t^s \eta^i j \partial_t \rho_0^2 J^{-2} A_t^k \partial_t dx dt (7.44)

We first consider the double integral (7.42). For the case $s = 0$, we write it as

\[
\int_0^T \int_\Omega \partial_t^{2r} \eta^i j \partial_t^{2r} (J^{-2} A_t^k) \partial_t^s \eta^i j \partial_t \rho_0^2 J^{-2} A_t^k \partial_t dx dt = \sum_{m=0}^{2r} C_{2m}^r \int_0^T \int_\Omega \partial_t^{2r} \eta^i j \partial_t^m J^{-2} \partial_t^{2r-m} A_t^k \partial_t \rho_0^2 J^{-2} A_t^k \partial_t dx dt.

For $m = 0, 3, 2r - 1, 2r$, we can use the Hölder inequality, (7.15) and the Sobolev embedding theorem to get the desired bound. In particular, we have to use an $L^2 - L^3 - L^6$ Hölder inequality and (7.15) to deal with the integral involving the terms of the form $\partial_t^{2r} \nabla \eta \partial_t^{2r-3} \nabla \eta \partial_t^3 \nabla \eta$ in order to get the bound. For $m = 1, 2, 4, \cdots, 2r - 2$, we can only apply the Hölder inequality and the Sobolev embedding theorem to get the desired bound by noticing that $\rho_0 \partial_t^{2r} J^{-2} \in H^{r-\ell}(\Omega)$ for $0 \leq \ell < r - 1$.

For the case $s = 1$, since $v_t \in H^{r-1}(\Omega)$, or $\nabla v_t \in L^\infty(\Omega)$, it is similar to and easier than those of the case $s = 0$. We omit the details.

For the case $s = 2$, we have

\[
\int_0^T \int_\Omega \partial_t^{2r} \eta^i j \partial_t^{2r-2} (J^{-2} A_t^k) \partial_t^2 \eta^i j \partial_t \rho_0^2 J^{-2} A_t^k \partial_t dx dt

= \sum_{m=0}^{2r-2} C_{2m}^r \int_0^T \int_\Omega \partial_t^{2r} \eta^i j \partial_t^m J^{-2} \partial_t^{2r-m} A_t^k \partial_t^2 \eta^i j \partial_t \rho_0^2 J^{-2} A_t^k \partial_t dx dt.

For $m = 0$, we must use $E_5(t)$ to control $\|\partial_t^{2r} \nabla \eta\|_{L^\infty}(\Omega)$ when $r = 4$; while it is easy to get the desired bound for $r = 5$. For $m = 1$, we can use an $L^2 - L^3 - L^6$ Hölder inequality and (7.15) to obtain the desired bound, i.e., $M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t))$. For $m = 2, \cdots, 2r -$
2, they are controlled by the desired bounds by using the Hölder inequality and the Sobolev embedding theorem.

For the case \( s = 3 \), we get
\[
\int_0^T \int_\Omega \partial_t^{2r} \eta_{i,j} \partial_t^{2r-3}(J^{-2}A_t^k) \partial_t^3 \mathbf{v}_{j,k} \rho_0^2 JA^l_i dx dt
\]
\[
\sum_{m=0}^{2r-3} C_{2r-3}^m \int_0^T \int_\Omega \partial_t^{2r} \eta_{i,j} \partial_t^{m} J^{-2} \partial_t^{2r-3-m} A_t^k \partial_t^3 \mathbf{v}_{j,k} \rho_0^2 JA^l_i dx dt.
\]

For \( m = 0 \), we can use an \( L^2-L^3-L^6 \) Hölder inequality and (7.15) to obtain the desired bound. For \( m = 1, \ldots, 2r-3 \), they are controlled by the desired bounds by using the Hölder inequality and the Sobolev embedding theorem.

For the case \( s = 4 \), we can use an \( L^2-L^6-L^3 \) Hölder inequality, and (7.15) for \( r = 4 \) additionally, to obtain the desired bound. For the cases \( s = 5, \ldots, 2r-2 \), we can use the Hölder inequality and the Sobolev embedding theorem to get the desired bounds.

Next, we consider the integral (7.43). For the case \( s = 0 \), we have
\[
\int_0^T \int_\Omega \partial_t^{2r} \eta_{i,j} \partial_t^{2r-1}(J^{-2}A_t^k) \partial_t^3 \mathbf{v}_{j,k} \rho_0^2 JA^l_i dx dt
\]
\[
\sum_{m=0}^{2r-1} C_{2r-1}^m \int_0^T \int_\Omega \partial_t^{2r} \eta_{i,j} \partial_t^{m} J^{-2} \partial_t^{2r-1-m} A_t^k \partial_t^3 \mathbf{v}_{j,k} \rho_0^2 JA^l_i dx dt.
\]

which can be controlled by the desired bound by using the Hölder inequality and the Sobolev embedding theorem.

For the case \( s = 1 \), it follows that
\[
\int_0^T \int_\Omega \partial_t^{2r} \eta_{i,j} \partial_t^{2r-2}(J^{-2}A_t^k) \partial_t^3 \mathbf{v}_{j,k} \rho_0^2 JA^l_i dx dt
\]
\[
\sum_{m=0}^{2r-2} C_{2r-2}^m \int_0^T \int_\Omega \partial_t^{2r} \eta_{i,j} \partial_t^{m} J^{-2} \partial_t^{2r-2-m} A_t^k \partial_t^3 \mathbf{v}_{j,k} \rho_0^2 JA^l_i dx dt.
\]

For \( m = 0 \), we can use the fact \( v \in H^4(\Omega) \), which is contained in \( E_5(t) \), for all cases \( r = 4, 5 \) to obtain
\[
\left| \int_0^T \int_\Omega \partial_t^{2r} \eta_{i,j} J^{-2} \partial_t^{2r-2} A_t^k \partial_t^3 \mathbf{v}_{j,k} \rho_0^2 JA^l_i dx dt \right| \leq M_0 + \delta \sup_{[0,T]} E_5(t) + CTP \left( \sup_{[0,T]} E_5(t) \right).
\]

For \( m = 1 \), we can use an \( L^2-L^3-L^6 \) Hölder inequality, (7.15) and the Sobolev embedding theorem to get the desired bound. For other cases of \( m \), we can use the Hölder inequality and the Sobolev embedding theorem to get the desired bound by noticing that \( \rho_0^2 \partial_t^{2\ell} J^{-2} \in H^{r-\ell}(\Omega) \) for \( 0 \leq \ell \leq r-1 \) with the help of the fundamental theorem of calculus if necessary.

For the case \( s = 2 \), we get
\[
\int_0^T \int_\Omega \partial_t^{2r} \eta_{i,j} \partial_t^{2r-3}(J^{-2}A_t^k) \partial_t^3 \mathbf{v}_{j,k} \rho_0^2 JA^l_i dx dt
\]
\[
\sum_{m=0}^{2r-3} C_{2r-3}^m \int_0^T \int_\Omega \partial_t^{2r} \eta_{i,j} \partial_t^{m} J^{-2} \partial_t^{2r-3-m} A_t^k \partial_t^3 \mathbf{v}_{j,k} \rho_0^2 JA^l_i dx dt,
\]

which can be controlled by the desired bound by using the Hölder inequality and the Sobolev embedding theorem, in addition, with the help of (7.15) for \( m = 0 \).

For other cases of \( s \), we can use similar argument to get the desired bounds and omit the details.
For the spatial integral \((7.41)\), we can use the same argument as for \((7.43)\) to get the desired bound with the help of the fundamental theorem of calculus. For the double integral \((7.44)\), it is easier to get the bound than either \((7.42)\) or \((7.43)\) and thus we omit the details. Therefore, we obtain the estimates for \(\int_0^T (7.4)\)dt, i.e.,

\[
\left| \int_0^T (7.4) \right| dt \leq M_0 + \delta \sup_{[0,T]} E_5(t) + CTP(\sup_{[0,T]} E_5(t)).
\]

**Step 5. Analysis of the remainder \(\int_0^T \mathcal{I}_3 dt\).** By integration by parts with respect to the spatial variables and the time variable, respectively, we obtain

\[
\int_0^T \mathcal{I}_3 dt = 2^{r-1} \sum_{i=1}^{2r-1} C_{2r} \int_0^T \partial_t^{2r-i} \partial_i J^{-2} \partial_t^{2r} \eta_i \rho_0^2 dx dt
\]

\[
= 2^{r-1} \sum_{i=1}^{2r-1} C_{2r} \int_0^T \partial_t^{2r-i} \partial_i J^{-2} \partial_t^{2r} \eta_i \rho_0^2 |^T_0
\]

\[
- \sum_{i=1}^{2r-1} C_{2r} \int_0^T \partial_t^{2r-i} \partial_i J^{-2} \partial_t^{2r} \eta_i \rho_0^2 dx dt
\]

\[
- \sum_{i=1}^{2r-1} C_{2r} \int_0^T \partial_t^{2r-i} \partial_i J^{-2} \partial_t^{2r} \eta_i \rho_0^2 dx dt,
\]

due to \(\partial_t^{2r-i} a_i^3 \partial_t^{2r} v = 0\) on \(\Gamma_0\).

We first consider \((7.46)\). For the case \(l = 1\),

\[
\int_0^T \int_\Omega \partial_t^{2r} \partial_t a_i^3 \partial_i J^{-2} \partial_t^{2r} \eta_i \rho_0^2 dx dt
\]

\[
= \int_0^T \int_\Omega \partial_t^{2r-1} (\partial_t \eta_i p) a_i^3 \partial_i J^{-1} \partial_t^{2r} \eta_i \rho_0^2 dx dt
\]

\[
= \sum_{s=1}^{2r-1} C_{2r-1} \int_0^T \int_\Omega \partial_t^{2r-s} \eta_i p a_i^3 \partial_t^{s} \partial_t J^{-1} \partial_t^{2r} \eta_i \rho_0^2 dx dt
\]

\[
+ \int_0^T \int_\Omega \partial_t^{2r} \eta_i p a_i^3 \partial_t^{2r} \eta_i \partial_t \rho_0^2 dx dt.
\]

Since \(\partial_t J^{-2} \in L^\infty(\Omega)\), we can use a similar argument as in \((7.3)\) to get the estimates of \((7.48)\).

For \((7.49)\), we easily have

\[
| (7.49) | \leq \epsilon T \sup_{[0,T]} \| a_i^3 \partial_t^2 \nabla \eta_i \|_0^2 \| \nabla v \|_2 P(E_4(t)) \leq CTP(\sup_{[0,T]} E_4(t)).
\]

For the case \(l = 2\), similar to the case \(l = 1\), we can get the bound easily since \(\partial_t^2 J^{-2} \in L^\infty(\Omega)\) and we omit the details. For the case \(l = 3\), we get

\[
\int_0^T \int_\Omega \partial_t^{2r-2} \partial_t a_i^3 \partial_i J^{-2} \partial_t^{2r} \eta_i \partial_t \rho_0^2 dx dt
\]

\[
= \int_0^T \int_\Omega \partial_t^{2r-3} (\partial_t \eta_i p) a_i^3 \partial_i J^{-1} \partial_t^{2r} \eta_i \partial_t \rho_0^2 dx dt
\]

\[
= \sum_{s=1}^{2r-3} C_{2r-3} \int_0^T \int_\Omega \partial_t^{2r-3-s} \eta_i p a_i^3 \partial_t^{s} \partial_t J^{-1} \partial_t^{2r} \eta_i \partial_t \rho_0^2 dx dt
\]

\[
+ \int_0^T \int_\Omega \partial_t^{2r-2} \eta_i p a_i^3 \partial_t^{2r} \eta_i \partial_t \rho_0^2 \partial_t J^{-1} \partial_t a_i^3 \partial_t \rho_0^2 dx dt.
\]
In (7.50), we use an $L^3-L^6-L^2$ Hölder inequality and (7.15) for the higher order terms of the cases $s=1$ and $s=2r-3$ and an $L^6-L^b-L^b$ Hölder inequality for the other cases to get the desired bounds. For (7.51), since $\rho_0 \partial_t^4 J^{-2} \in H^{r-2}(\Omega) \subset L^\infty(\Omega)$, we can get the desired bound easily in view of the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus. For the case $l=4$, we have the desired bound as a similar argument as for the case $l=3$. For the case $l=2r-3$, we can use an $L^6-L^2-L^2$ Hölder inequality and (7.15) to get the desired bound. For the case $l=2r-2$, we use an $L^3-L^6-L^2$ Hölder inequality and the Sobolev embedding theorem to get the bound due to $\rho_0 \partial_t^{2(r-1)} J^{-2} \in H^1(\Omega) \subset L^6(\Omega)$. For the case $l=2r-1$, it is similar to the case $s=1$ in (7.42) and we omit the details. For the other cases, we can easily get the desired bounds by using the Hölder inequality and the Sobolev embedding theorem.

Next, we consider (7.47). Since the cases $1 \leq l \leq 2r-2$ are identical to the cases $2 \leq l \leq 2r-1$ of (7.46) estimated just discussed up to some constant multipliers, we only need to consider the remainder case $l=2r-1$. We can apply (1.11) to split the integral of the case $l=2r-1$ into two integrals. One of them can be used an $L^2-L^2$ Hölder inequality to get the estimates, the other one can be dealt with as the same arguments as for the case $l=2r-1$ of (7.46) or the case $s=1$ in (7.42). Thus, we omit the details.

For the spatial integral (7.45), it can be estimated as the same arguments as for (7.46) or (7.47) with the help of the fundamental theorem of calculus whose details are omitted.

**Step 6. Summing inequalities.** As the same argument as in the estimates of the horizontal derivatives, we can obtain the desired result by combining the previous inequalities. □

8. The Estimates for the Mixed Time-Horizontal Derivatives

We have the following estimates.

**Proposition 8.1.** Let $r \in \{4,5\}$ and $1 \leq m \leq r-1$. For $\delta > 0$ and the constant $M_0$ depend on $1/\delta$, we have

$$\sup_{[0,T]} \left[ \|\rho_0^{1/2} \partial_t^2 \overline{d}^{r-m} v\|_0^2 + \|\rho_0 \nabla \partial_t^2 \overline{d}^{r-m} \eta\|_0^2 + \|\rho_0\text{div} \partial_t^2 \overline{d}^{r-m} \eta\|_0^2 + \|\partial_t^2 \overline{d}^{r-m} \eta\|_0^2 \right] 
\leq M_0 + \delta \sup_{[0,T]} E_r(t) + C \text{TP}(\sup_{[0,T]} E_r(t)).$$

**Proof.** Applying the differential operator $\partial_t^2 \overline{d}^{r-m}$ on (1.13a) and taking the $L^2(\Omega)$-inner product with $\partial_t^2 \overline{d}^{r-m} v^i$, we have, by integration by parts, (1.13b) and (1.13c), that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho_0 |\partial_t^2 \overline{d}^{r-m} v|^2 dx + \mathcal{J}_1 + \mathcal{J}_2 = \mathcal{J}_3,$$

where

$$\mathcal{J}_1 := - \int_{\Omega} \partial_t^2 \overline{d}^{r-m} v^i \partial_t^{2m} \overline{d}^{r-m} v^j \rho_0 J^{-2} dx,$$
$$\mathcal{J}_2 := - \int_{\Omega} \rho_0 \partial_t^2 \overline{d}^{r-m} v^i \partial_t^{2m} \overline{d}^{r-m} v^j dx,$$
$$\mathcal{J}_3 := - \sum_{i=0}^{r-m-1} C_{r-m}^i \int_{\Omega} \partial_t^{2m-1} \rho_0 \partial_t^{r-m-l} \partial_t^{2m-l} \partial_t^{2m} \overline{d}^{r-m} v^i \partial_t^{-k} \partial_t^l \overline{d}^{r-m} v^j dx \quad (8.1)$$
$$+ \sum_{s=1}^{2m} \sum_{l=0}^{r-m-1} \sum_{k=0}^{2m} C_{2m}^s C_{r-m}^l C_l^k \int_{\Omega} \partial_t^{l-k} \rho_0 \partial_t^{2m-1-l} \partial_t^{2m-s} \partial_t^{2m-k} \partial_t^{r-m} v^i \partial_t^{2m-l} \overline{d}^{r-m} v^j dx \quad (8.2)$$
$$+ \sum_{s=1}^{2m} \sum_{l=0}^{r-m-1} \sum_{k=0}^{2m} C_{2m}^s C_{r-m}^l \int_{\Omega} \partial_t^{l-k} \rho_0 \partial_t^{2m-s} \partial_t^{2m-k} \partial_t^{r-m} v^i \partial_t^{2m-1-l} \overline{d}^{r-m} v^j dx \quad (8.3)$$
Thus, we get
\begin{align}
\mathcal{J}_1 &= -\int_\Omega \partial_t^2 \partial_{\tau}^m \eta^{i,j} \partial_t \partial_{\tau}^m \eta^{i,j} \rho_0^2 J^{-1} \left( A^q_i A^q_j - A^q_i A^q_j \right) dx \\
&\quad - \int_\Omega \nabla \cdot \left( J \left( A^q_i A^q_j - A^q_i A^q_j \right) \right) \partial_t \partial_{\tau}^m \eta^{i,j} \rho_0^2 J^{-2} dx \\
&\quad - \frac{1}{2} \frac{d}{dt} \int_\Omega \partial_t^2 \partial_{\tau}^m \eta^{i,j} \partial_t \partial_{\tau}^m \eta^{i,j} \rho_0^2 J^{-1} dx + (8.5) \\
\mathcal{J}_2 &= 2 \int_\Omega \partial_t \partial_{\tau}^m \eta^{i,j} \partial_t \partial_{\tau}^m \eta^{i,j} \rho_0^2 J^{-1} A^q_d dx \\
&\quad + 2 \int_\Omega \partial_t \partial_{\tau}^m \eta^{i,j} \partial_t \partial_{\tau}^m \eta^{i,j} \rho_0^2 J^{-1} A^q_d dx + (8.7) \\
&\quad - \frac{d}{dt} \int_\Omega \left| \nabla \eta \partial_t \partial_{\tau}^m \eta \right|^2 \rho_0^2 J^{-1} dx + (8.7) \\
&\quad - \int_\Omega \partial_t \partial_{\tau}^m \eta^{i,j} \partial_t \partial_{\tau}^m \eta^{i,j} \rho_0^2 J^{-1} dx + (8.6) + (8.7) + (8.8).
\end{align}

By (11.1), we have
\begin{align}
\mathcal{J}_1 + \mathcal{J}_2 &= \frac{d}{dt} \int_\Omega \left[ \left| \nabla \eta \partial_t \partial_{\tau}^m \eta \right|^2 + \left| \nabla \cdot \left( J \partial_t \partial_{\tau}^m \eta \right) \right|^2 - \left| \nabla \eta \partial_t \partial_{\tau}^m \eta \right|^2 \right] \rho_0^2 J^{-1} dx \\
&\quad + (8.5) + (8.6) + (8.7) + (8.8).
\end{align}

Now, we analyze the integration with respect to time of the remainder integrals (8.1)-(8.8) and \( \mathcal{J}_4 \).

By the higher order Hardy inequality, the Hölder inequality, we have
\begin{align}
\left| \int_0^T (8.1) dt \right| &\leq C \left\| \rho_0 \right\|_{L^1}^2 \left\| \rho_0 \right\|_{L^2}^{1/2} \sup_{[0,T]} \left\| \rho_0 \partial_t^l \partial_{\tau}^m \nabla \eta \right\|_0 \left\| \rho_0 \partial_t^l \partial_{\tau}^m \eta \right\|_0 \\
&\leq M_0 + \delta \sup_{[0,T]} \left\| \rho_0 \partial_t^l \partial_{\tau}^m \eta \right\|_0^2 + C T P \left( \sup_{[0,T]} E_r (t) \right).
\end{align}

As a similar arguments as for the remainder integrals in Section 6, we can get the integrations over \([0,T]\) of (8.2)-(8.8) can be bounded by
\begin{align}
M_0 + \delta \sup_{[0,T]} E_r (t) + C T P \left( \sup_{[0,T]} E_r (t) \right).
\end{align}

Therefore, we can get the desired estimates by similar arguments as in Sections 6 and 7.

Finally, we show the boundary estimates. By Lemma 2.4, (2.5), the fundamental theorem of calculus, Hölder’s inequality and curl estimates, we obtain
\begin{align}
\left| \partial_t \partial_{\tau}^m \eta \right|_{1/2}^2 \leq \left\| \partial_{\tau}^m \partial_t \partial_{\tau}^m \eta \right\|_0^2 + \left\| \nabla \partial_t \partial_{\tau}^m \eta \right\|_0 \\
\lesssim \left\| \rho_0 \partial_t \partial_{\tau}^m \eta \right\|_0^2 + \left\| \rho_0 \nabla \partial_t \partial_{\tau}^m \eta \right\|_0^2 + \left\| \nabla \partial_t \partial_{\tau}^m \eta \right\|_0^2
\end{align}
Thus, we complete the proof. □

9. The elliptic-type estimates for the normal derivatives

Our energy estimates provide a priori control of horizontal and time derivatives of $\eta$: it remains to gain a priori control of the normal derivatives of $\eta$. This is accomplished via a bootstrapping procedure relying on the fact $\partial_i^\beta v(t)$ is bounded in $L^2(\Omega)$.

**Proposition 9.1.** For $t \in [0, T]$, it holds that
\[
\sup_{[0,T]} \left( \| \partial_i^\beta v(t) \|_1^2 + \| \partial_0 \partial_i^{8}\partial_i^\beta J^{2-}(t) \|_1^2 \right) \leq M_0 + \delta \sup_{[0,T]} E(t) + CT P(\sup_{[0,T]} E(t)).
\]

**Proof.** From (1.13a), we have for $\beta = 1, 2$
\[
\rho_0 a_i^3 J^{-2,3} + 2a_i^3 \rho_0 J^{-2} = -v_i - \rho_0 \partial_i^\beta J^{-2,\beta} - 2a_i^\beta \rho_0 J^{-2}. \tag{9.1}
\]

Acting $\partial_i^8$ on (9.1), we get
\[
\rho_0 a_i^3 \partial_i^8 J^{-2,3} + 2a_i^3 \rho_0 \partial_i^8 J^{-2} = -\partial_i^9 v_i - \rho_0 \partial_i^8 (a_i^\beta J^{-2,\beta}) - 2\rho_0 \partial_i^8 (a_i^\beta J^{-2})
\]
\[
- \partial_i^8 a_i^3 [\rho_0 J^{-2,3} + 2\rho_0 J^{-2} - \sum_{l=1}^7 C_8 l^{\delta_l} \partial_i^3 [\rho_0 J^{-2,3} + 2\rho_0 J^{-2}]].
\]

By (2.5), the fundamental theorem of calculus and Hölder’s inequality, we have
\[
\| \partial_i^9 v(t) \|_0^2 \leq C \int_\Omega \rho_0^2 \left( |\partial_i^9 v|^2 + |\nabla \partial_i^9 v|^2 \right) dx
\]
\[
\leq C \int_\Omega \rho_0^2 \left( \int_0^t |\partial_i^{10} v(t') + \partial_i^9 v(0) |^2 dt' + C \int_\Omega \rho_0^2 |\partial_i^9 v(0) |^2 dx + C \| \partial_i^9 \nabla v \|_0^2 \right)
\]
\[
\leq Ct \int_\Omega \rho_0^2 \int_0^t |\partial_i^{10} v|^2 dt' + C \int_\Omega \rho_0^2 |\partial_i^9 v(0) |^2 dx + C \| \partial_i^8 \nabla v \|_0^2
\]
\[
\leq C t^2 \| \rho_0 \|_{L^\infty(\Omega)} \sup_{[0,t]} \| \rho_0^{1/2} \partial_i^{10} v \|_0^2 + \| \rho_0 \|_{L^\infty(\Omega)} \| \partial_i^9 v(0) \|_0^2 + C \| \rho_0 \partial_i^8 \nabla v \|_0^2.
\]

By the fundamental inequality of algebra, the fundamental theorem of calculus, the Hölder inequality and the Sobolev embedding theorems, we see that
\[
\| \rho_0 \partial_i^8 (a_i^\beta J^{-2,\beta}) \|_0^2 \leq C \sum_{l=0}^4 \| \rho_0 \partial_i^{8-2l} a_i^\beta \partial_i^{2l} J^{-2,\beta} \|_0^2
\]
\[
\quad + C \sum_{l=1}^4 \| \rho_0 \partial_i^{8-2l} a_i^\beta \left( \int_0^t \partial_i^{2l} J^{-2,\beta} dt' + \partial_i^{2l} J^{-2,\beta}(0) \right) \|_0^2
\]
\[
\leq M_0 + Ct^2 \sup_{[0,t]} \sum_{l=0}^4 \| \rho_0 \partial_i^{2l} J^{-2} \|_{\mathcal{S}_{l+1}}^2 P(\| \partial_i^{8-2l} \eta \|_{l+1, \cdots} \| \eta \|_4, \| \eta \|_4, \| \eta \|_5).
Similarly, we also have that

\[ \| \partial_i^3 a_i \rho_0 J^{-2} \|_0 \leq M_0 + C t^2 \sup_{[0,t]} \| \rho_0 J^{-2} \|^3 \frac{1}{3} P(\| \partial_i^8 \eta \|_1, \cdots, \| \eta_t \|_4, \| \eta_t \|_4, \| \eta \|_5), \]

and

\[ \| \rho_{0,i} \partial_i^8 (a_i^8 J^{-2}) \|_0^2 \leq M_0 + C t^2 \left( \frac{\rho_{0,i}}{\rho_0} \right)^2 \sup_{[0,t]} \sum_{i=0}^4 \| \rho_0 \partial_i^2 J^{-2} \|^3 \frac{1}{3} P(\| \partial_i^8 \eta \|_1, \cdots, \| \eta_t \|_4, \| \eta_t \|_4, \| \eta \|_5). \]

By the fundamental theorem of calculus, we get

\[ \| \partial_i^8 a_i^3 \rho_0 J^{-2} \|_0^2 \leq M_0 + C t^2 \| \rho_{0,3} \|^3 \frac{1}{3} P(\| \partial_i^8 \eta \|_1, \cdots, \| \partial_i^8 \eta \|_1, \cdots, \| \eta_t \|_4, \| \eta_t \|_4, \| \eta \|_5). \]

Similarly, it follows that

\[ \sum_{i=1}^7 \| \partial_i^{8-i} a_i^3 \partial_i^l [\rho_0 J^{-2},3 + 2 \rho_{0,3} J^{-2}] \|_0^2 \leq M_0 + C t^2 \sup_{[0,t]} \sum_{l=0}^4 \| \rho_0 \partial_i^2 J^{-2} \|^3 \frac{1}{3} P(\| \partial_i^8 \eta \|_1, \cdots, \| \eta_t \|_4, \| \eta_t \|_4, \| \eta \|_5) \]

\[ + C t^2 \| \rho_{0,3} \|^3 \frac{1}{3} P(\| \partial_i^8 \eta \|_1, \cdots, \| \partial_i^8 \eta \|_1, \cdots, \| \eta_t \|_4, \| \eta_t \|_4, \| \eta \|_5). \]

Thus, we have obtained, for all \( t \in [0,T] \), that

\[ \| \rho_0 a_i^3 \partial_i^8 J^{-2,3} + 2 a_i^3 \rho_{0,3} \partial_i^8 J^{-2} \|_0^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + C T P(\sup_{[0,T]} E(t)). \]

It follows that

\[ \| \rho_0 a_i^3 \partial_i^8 J^{-2,3} \|_0^2 + 4 \| a_i^3 \rho_{0,3} \partial_i^8 J^{-2} \|_0^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + C T P(\sup_{[0,T]} E(t)) - 4 \int_\Omega \rho_0 \rho_{0,3} |a_i^3|^2 \partial_i^8 J^{-2,3} \partial_i^8 J^{-2} dx. \]

By Hölder’s inequality and the fundamental theorem of calculus, we get

\[ \left| -4 \int_\Omega \rho_0 \rho_{0,3} |a_i^3|^2 \partial_i^8 J^{-2,3} \partial_i^8 J^{-2} dx \right| \]

\[ \leq M_0 + \delta \| \rho_0 \partial_i^8 J^{-2,3} \|_0^2 + \delta \| \rho_{0,3} \partial_i^8 J^{-2} \|_0^2 + C T P(\sup_{[0,T]} E(t)), \]

and then, by the fundamental theorem of calculus once again,

\[ \| \rho_0 \partial_i^8 J^{-2,3} \|_0^2 + \| \rho_{0,3} \partial_i^8 J^{-2} \|_0^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + C T P(\sup_{[0,T]} E(t)). \]

By integration by parts with respect to \( x_3 \), the Hölder inequality and noticing that \( \rho_0 = 0 \) on \( \Gamma_1 \), it holds

\[ \| \rho_0 \partial_i^8 J^{-2}(t) \|_0^2 = \int_\Omega \partial_3 x_3 (\rho_0 \partial_i^8 J^{-2})^2 dx \]

\[ = -2 \int_\Omega x_3 \rho_0 \partial_i^8 J^{-2} (\rho_{0,3} \partial_i^8 J^{-2} + \rho_0 \partial_i^8 J^{-2,3}) dx \]

\[ \leq 2 \| \rho_0 \partial_i^8 J^{-2}(t) \|_0 \left[ \| \rho_{0,3} \partial_i^8 J^{-2} \|_0 + \| \rho_0 \partial_i^8 J^{-2,3} \|_0 \right], \]
which implies that
\[
\|\rho_0 \partial_t^8 J^{-2}(t)\|_0^2 \leq 8 \left[ \|\rho_{0,3} \partial_t^8 J^{-2}\|_0^2 + \|\rho_0 \partial_t^8 J^{-2,3}\|_0^2 \right] \\
\leq M_0 + \delta \sup_{[0,T]} E(t) + CTP\left(\sup_{[0,T]} E(t)\right). \tag{9.2}
\]
Since we can get, by Proposition 8.1, that
\[
\|\rho_0 \partial_t^8 J^{-2}\|_0^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP\left(\sup_{[0,T]} E(t)\right),
\]
we have
\[
\|\rho_0 \partial_t^8 J^{-2}\|_1^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP\left(\sup_{[0,T]} E(t)\right).
\]
It follows from (2.5) and (9.2) that
\[
\|\partial_t^8 J^{-2}\|_0^2 \leq C|\|\rho_0 \partial_t^8 J^{-2}\|_0^2 + C|\|\rho_0 \nabla \partial_t^8 J^{-2}\|_0^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP\left(\sup_{[0,T]} E(t)\right).
\]
Due to (1.11), we see that
\[
\partial_t^8 J^{-2} = -2 \partial_t^7 (J^{-2} A^i_i v^j, j) = -2 J^{-2} A^i_i \partial_t^7 v^j, j - 2 \partial_t^6 (J^{-2} A^i_i) v^j, j - 2 \sum_{i=1}^6 C_i \partial_t^6 (J^{-2} A^j_j) \partial_t^7 \bar{\nu}, j,
\]
namely, in view of the fundamental theorem of calculus,
\[
\text{div} \partial_t^7 v = -\frac{1}{2} \partial_t^8 J^{-2} - \partial_t^7 v^j, j \int_0^t (J^{-2} A^i_i), dt' - \partial_t^7 (J^{-2} A^i_i) v^j, j - \sum_{i=1}^6 C_i \partial_t^6 (J^{-2} A^j_j) \partial_t^7 \bar{\nu}, j.
\]
We can easily estimate last three terms by using the fundamental theorem of calculus and the H"{o}lder inequality. Thus, we obtain
\[
\|\text{div} \partial_t^7 v\|_0^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP\left(\sup_{[0,T]} E(t)\right).
\]
According to Proposition 5.1, we have
\[
\|\text{curl} \partial_t^7 v\|_0^2 \leq M_0 + CTP\left(\sup_{[0,T]} E(t)\right).
\]
With the boundary estimates on $\partial_t^7 v^\alpha$ or $\partial_t^8 \eta^\alpha$ given by Proposition 8.1, we obtain, from Lemma 2.5, that
\[
\|\partial_t^7 v\|_1^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP\left(\sup_{[0,T]} E(t)\right).
\]
Thus, we complete the proof. \qed

**Proposition 9.2.** For $t \in [0, T]$, it holds that
\[
\sup_{[0,T]} \left[ \|\partial_t^7 v(t)\|_2^2 + \|\rho_0 \partial_t^6 J^{-2}(t)\|_2^2 \right] \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP\left(\sup_{[0,T]} E(t)\right).
\]
**Proof.** Applying the differential operator $\partial_t^6$ on (9.1), we have
\[
\rho_0 \partial_t^3 \partial_t^6 J^{-2,3} + 2 \rho_0 \partial_t^3 \partial_t^6 J^{-2} = -\partial_t^7 v^i - \rho_0 \partial_t^6 (a^i_j J^{-2,\beta}) - 2 \rho_0 \beta \partial_t^6 (a^i_j J^{-2}) - \partial_t^6 \partial_t^3 \left[ \rho_0 J^{-2,3} + 2 \rho_0 \partial_t^3 J^{-2} \right] \\
- \sum_{i=1}^5 C_i \partial_t^6 (J^{-2,\beta} + 2 \rho_0 \partial_t^3 J^{-2}). \tag{9.3}
\]
We first estimate horizontal derivatives of \( \rho_{0,3} \partial_t^6 J^{-2}(t) \) in \( L^2(\Omega) \) to consider for \( \alpha = 1, 2 \),
\[
\rho_0 a_3^3 \partial_t^6 J^{-2,3\alpha} + 2a_3^3 \rho_{0,3} \partial_t^6 J^{-2,\alpha}
\]
\[
= \left[ -\partial_t^7 v - \rho_0 \partial_t^6 (a_t^6 J^{-2,\beta}) - 2\rho_0 \partial_t^6 (a_t^6 J^{-2}) - \partial_t^6 a_3^3 [\rho_0 J^{-2,3} + 2\rho_{0,3} J^{-2}] \right]
\]
\[
- \sum_{l=1}^{5} C_l^6 \partial_t^6 - l^3 a_l^3 \partial_t^l [\rho_0 J^{-2,3} + 2\rho_{0,3} J^{-2}] \right] \]
\[
- 2\rho_{0,3} a_t^3 \partial_t^6 J^{-2} - 2a_t^3 \rho_{0,3} \partial_t^6 J^{-2}.
\]
Now, we estimate the \( L^2(\Omega) \) norms of the right hand side. From Proposition 9.1, we know that
\[
\| \partial_t^7 \tilde{\mathbf{a}} \|_0^2 \leq \| \partial_t^7 v \|_1^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)).
\]
Since
\[
\left\| \rho_0 \partial_t^6 (a_t^6 J^{-2,\beta}) \right\|_0^2 \leq \| \rho_0 \partial_t^6 (a_t^6 J^{-2,\beta}) \|_0^2 \leq \| \rho_0 \partial_t^6 (J^{-1} A_t^6 A_t^6 \eta_k, \beta) \|_0^2 + \| \rho_0 \partial_t^6 (J^{-1} A_t^6 A_t^6 \eta_k, \beta) \|_0^2,
\]
we consider the last term involving the highest order derivatives
\[
\partial_t^6 (J^{-1} A_t^6 A_t^6 \eta_k, \beta, \alpha) = \partial_t^6 [ -J^{-1} A_t^6 \eta_t^{\alpha, \beta} A_t^6 A_t^6 \eta_k, \beta + J^{-1} A_t^6 \eta_t^{\alpha, \beta} A_t^6 A_t^6 \eta_k, \beta ]
\]
For the highest order derivatives term, by the fundamental theorem of calculus, Sobolev’s embedding theorem and Proposition 8.1, we have for \( T > 0 \) small enough that
\[
\| \rho_0 J^{-1} A_t^6 A_t^6 \partial_t^6 \eta, \beta, \alpha \|_0^2 \leq \| \rho_0 \text{div} \partial_t^6 \tilde{\mathbf{a}} \eta \|_0^2 + \| \rho_0 \int_0^T \partial_t (J^{-1} A_t^6 A_t^6) dt \partial_t^6 \tilde{\mathbf{a}} \eta, \beta, \alpha \|_0^2
\]
\[
\leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t))
\]
\[
+ CTP(\sup_{[0,T]} \| \eta \|_5, \| \eta \|_5, \| \rho_0 \eta_t \|_0^2, \| \rho_0 \nabla \partial_t^6 \tilde{\mathbf{a}} \eta \|_0^2)
\]
\[
\leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)).
\]
For the lower order derivatives terms, we use a similar argument to get the bound. Thus, we can get
\[
\left\| \rho_0 \partial_t^6 (a_t^6 J^{-2,\beta}) \right\|_0^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)).
\]
We can deal with the remainder terms in the right hand side of (9.4) by the same argument to get the bound in \( L^2(\Omega) \) and thus we have
\[
\| \rho_0 a_t^3 \partial_t^6 J^{-2,3\alpha} + 2a_t^3 \rho_{0,3} \partial_t^6 J^{-2,\alpha} \|_0^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)).
\]
It follows that
\[
\| \rho_0 a_t^3 \partial_t^6 J^{-2,3\alpha} \|_0^2 + 4\| a_t^3 \rho_{0,3} \partial_t^6 J^{-2,\alpha} \|_0^2
\]
\[
\leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)) - 4 \int \rho_0 a_t^3 (a_t^3)^2 \partial_t^6 J^{-2,3\alpha} \partial_t^6 J^{-2,\alpha} dx.
\]
By Hölder’s inequality and the fundamental theorem of calculus, we get
\[ -4 \int_\Omega \rho_0^{22-3} \rho_{0,3} |a^3|^2 \partial_t^6 J^{-2} \cdot_{3\alpha} \partial_t^6 J^{-2} \cdot_{\alpha} dx \]
\[ \leq M_0 + \delta \| \rho_0 \partial_t^6 J^{-2} \cdot_{3\alpha} \|_0^2 + \delta \| \rho_{0,3} \partial_t^6 J^{-2} \cdot_{\alpha} \|_0^2 + CTP(\sup E(t)), \]
and then, by the fundamental theorem of calculus once again,
\[ \| \rho_0 \partial_t^6 J^{-2} \cdot_{3\alpha} \|_0^2 + \| \rho_{0,3} \partial_t^6 J^{-2} \cdot_{\alpha} \|_0^2 \leq M_0 + \delta \sup E(t) + CTP(\sup E(t)). \]

Similar to (9.2), we get
\[ \| \rho_0 \partial_t^6 J^{-2} \cdot_{\alpha} \|_0^2 \leq M_0 + \delta \sup E(t) + CTP(\sup E(t)). \]

From Proposition 8.1, we can obtain that
\[ \| \rho_0 \partial_t^6 J^{-2} \cdot_{\alpha} \|_0^2 \leq M_0 + \delta \sup E(t) + CTP(\sup E(t)). \]

Thus, we have
\[ \| \rho_0 \partial_t^6 J^{-2} \cdot_{\alpha} \|_0^2 \leq \| \rho_0 \partial_t^6 J^{-2} \cdot_{\alpha} \|_0^2 + \frac{\partial \rho_0}{\rho_0} \|_{L^\infty(\Omega)}^2 \| \rho_0 \partial_t^6 J^{-2} \cdot_{\alpha} \|_0^2 + \| \rho_{0,3} \partial_t^6 J^{-2} \cdot_{\alpha} \|_0^2 \]
\[ \leq M_0 + \delta \sup E(t) + CTP(\sup E(t)), \]
and by using the inequality (2.5),
\[ \| \partial_t^6 J^{-2} \cdot_{\alpha} \|_0^2 \leq \| \rho_0 \partial_t^6 J^{-2} \cdot_{\alpha} \|_0^2 + \| \rho_0 \partial_t^6 J^{-2} \cdot_{\alpha} \|_0^2 \leq M_0 + \delta \sup E(t) + CTP(\sup E(t)). \]

Due to (1.11), we see that
\[ \partial_t^6 J^{-2} \cdot_{\alpha} = -2 \partial_t^6 (J^{-2} A^t J^{-1} v^t \cdot_{\alpha}), \]
\[ = -2 J^{-2} A^t \partial_t^5 v^t, j_{\alpha} - 2 (J^{-2} A^t) \partial_t^5 v^t, j_{\alpha} - 2 \partial_t^6 (J^{-2} A^t) \partial_t^5 v^t, j_{\alpha} \]
\[ - 2 \partial_t^5 (J^{-2} A^t) \partial_t^4 v^t, j_{\alpha} - 2 \sum_{l=1}^4 C_5 \partial_t^l (J^{-2} A^t) \partial_t^{5-l} v^t, j_{\alpha}, \]

namely, in view of the fundamental theorem of calculus,
\[ \text{div} \partial_t^5 v, \alpha = -\frac{1}{2} \partial_t^5 J^{-2} \cdot_{\alpha} - \int_0^t (J^{-2} A^t) dt' \partial_t^5 v, j_{\alpha} - (J^{-2} A^t) \partial_t^5 v, j_{\alpha} \]
\[ - \partial_t^5 (J^{-2} A^t) \partial_t^4 v, j_{\alpha} - \partial_t^5 (J^{-2} A^t) \partial_t^4 v, j_{\alpha} - \sum_{l=1}^4 C_5 \partial_t^l (J^{-2} A^t) \partial_t^{5-l} v^t, j_{\alpha}. \]

We can easily estimate last three terms by using the fundamental theorem of calculus and the Hölder inequality. Thus, we obtain
\[ \| \text{div} \partial_t^5 v, \alpha \|_0^2 \leq M_0 + \delta \sup E(t) + CTP(\sup E(t)). \]

According to Proposition 5.1, we have
\[ \| \text{curl} \partial_t^5 v \|_0^2 \leq \| \text{curl} \partial_t^5 v \|_1^2 \leq M_0 + CTP(\sup E(t)). \]
With the boundary estimates on $\partial^5 x \nu^\beta$ given by Proposition 8.1, we obtain, from Lemma 2.5, that
\[
\| \partial^5 x_{,\alpha} \|_1^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)).
\]

Thus, we have proved for $\alpha = 1, 2$
\[
\sup_{[0,T]} \left[ \| \partial^5 x_{,\alpha}(t) \|_1^2 + \| \rho_0 \partial_i^6 J^{-2,\alpha}(t) \|_1^2 \right] \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)). \tag{9.5}
\]

We next differentiate (9.3) in the vertical direction $x_3$ to obtain
\[
\begin{align*}
\rho_0 a_i^3 \partial_i^6 J^{-2,33} + 3 \rho_0 a_i^3 \partial_i^6 J^{-2,3} = & \left[- \partial_j^3 v^i - \rho_0 \partial_i^6 (a_i^\beta J^{-2,\beta}) - 2 \rho_0 a_\beta \partial_i^6 (a_i^\beta J^{-2}) - \partial_i a_i^3 [\rho_0 J^{-2,3} + 2 \rho_0 J^{-2}] \right. \\
& - \sum_{l=3}^{6} C_i^{\lambda} \partial_i^{6-l} a_i^3 \partial_i^6 [\rho_0 J^{-2,3} + 2 \rho_0 J^{-2}] \right]_{,3} - \rho_0 a_i^3 \partial_i^6 J^{-2,3} \\
& - 2 \rho_0 a_3^3 a_i^3 \partial_i^6 J^{-2} - 2 a_i^3 \rho_0 a_i^3 \partial_i^6 J^{-2}. \tag{9.6}
\end{align*}
\]

Propositions 9.1 and 8.1 together with the inequality (9.5) show that the right hand side of (9.6) is bounded in $L^2(\Omega)$ by $M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t))$. It follows that
\[
\| \rho_0 a_i^3 \partial_i^6 J^{-2,33} + 3 \rho_0 a_i^3 \partial_i^6 J^{-2,3} \|_0^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)).
\]

Thus, by the Hölder inequality and the fundamental theorem of calculus, we get
\[
\begin{align*}
\| \rho_0 a_i^3 \partial_i^6 J^{-2,33} \|_0^2 & + \| \rho_0 a_i^3 \partial_i^6 J^{-2,3} \|_0^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)) \\
& - 6 \int_{\Omega} \rho_0 \rho_0 a_3^3 [\rho_0 J^{-2,3} \partial_3^6 J^{-2,33} \partial_3^6 J^{-2,33}] dx \\
& \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)) + \delta \| \rho_0 \partial_i^6 J^{-2,33} \|_0^2 + \delta \| \rho_0 \partial_i^6 J^{-2,3} \|_0^2,
\end{align*}
\]

and then, by the fundamental theorem of calculus once again,
\[
\| \rho_0 \partial_i^6 J^{-2,33} \|_0^2 + \| \rho_0 \partial_i^6 J^{-2,3} \|_0^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)).
\]

Similar to (9.2), we get, by integration by parts and Cauchy’s inequality, that
\[
\begin{align*}
\| \rho_0 \partial_i^6 J^{-2,3} \|_0^2 & = \int_{\Omega} \rho_0^2 (\partial_i^6 J^{-2,3})^2 dx \\
& = -2 \int_{\Omega} x_3 [\rho_0 \rho_0 a_3^3 (\partial_3^6 J^{-2,3})^2 + \partial_0^2 \partial_i^6 J^{-2,3} \partial_0 \partial_i^6 J^{-2,33}] dx \\
& \leq 2 \| \rho_0 \partial_i^6 J^{-2,3} \|_0 \| \rho_0 a_3^3 \partial_3^6 J^{-2,3} \|_0 + 2 \| \rho_0 \partial_i^6 J^{-2,3} \|_0 \| \rho_0 \partial_i^6 J^{-2,33} \|_0,
\end{align*}
\]

which implies, by eliminating one $\| \rho_0 \partial_i^6 J^{-2,3} \|_0$ from both sides and using the Cauchy inequality, that
\[
\| \rho_0 \partial_i^6 J^{-2,3} \|_0^2 \leq 8 \left( \| \rho_0 \partial_i^6 J^{-2,33} \|_0^2 + \| \rho_0 a_3^3 \partial_3^6 J^{-2,3} \|_0^2 \right) \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)).
\]

Then, by (9.5) and (2.5), we can obtain
\[
\| \rho_0 \partial_i^6 J^{-2} \|_0^2 + \| \partial_i^6 J^{-2} \|_0^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)).
\]
Thus, we can infer that
\[ \| \text{div} \partial_t^5 v(t) \|^2_{L^2} \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)). \]

According to Proposition 5.1, we have
\[ \| \text{curl} \partial_t^5 v \|^2 \leq M_0 + CTP(\sup_{[0,T]} E(t)). \]

With the boundary estimates on \( \partial_t^5 v^\beta \) given by Proposition 8.1, we obtain, from Lemma 2.5, that
\[ \| \partial_t^5 v \|^2 \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)). \]

This completes the proof. \( \Box \)

We also have the following propositions.

**Proposition 9.3.** For \( t \in [0, T] \), it holds that
\[ \sup_{[0,T]} \left[ \| v_{tt}(t) \|^2_{L^2} + \| \rho_0 \partial_t^4 J^{-2} (t) \|^2_{L^2} \right] \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)). \]

**Proof.** Applying \( \partial_t^4 \) on (9.1), we have
\[
\begin{align*}
\rho_0 a_i^3 \partial_t^4 J^{-2,3} + 2a_i^3 \rho_{0,3} \partial_t^4 J^{-2} \\
= -\partial_t^3 v^i - \rho_0 \partial_t^4 (a_i^\beta J^{-2,\beta}) - 2 \rho_{0,\beta} \partial_t^4 (a_i^\beta J^{-2}) \\
- \partial_t^4 a_i^3 \left[ \rho_0 J^{-2,3} + 2 \rho_{0,3} J^{-2} \right] - \sum_{l=1}^3 \partial_t^{4-l} a_i^3 \partial_t^l \left[ \rho_0 J^{-2,3} + 2 \rho_{0,3} J^{-2} \right].
\end{align*}
\]

The same argument used in the proof of Proposition 9.2 yields the desired results. \( \Box \)

**Proposition 9.4.** For \( t \in [0, T] \), it holds that
\[ \sup_{[0,T]} \left[ \| v(t) \|^2_{L^2} + \| \rho_0 \partial_t^2 J^{-2} (t) \|^2_{L^2} \right] \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)). \]

**Proof.** Applying \( \partial_t^2 \) on (9.1), we have
\[
\begin{align*}
\rho_0 a_i^3 \partial_t^2 J^{-2,3} + 2a_i^3 \rho_{0,3} \partial_t^2 J^{-2} \\
= -\partial_t^3 v^i - \rho_0 \partial_t^2 (a_i^\beta J^{-2,\beta}) - 2 \rho_{0,\beta} \partial_t^2 (a_i^\beta J^{-2}) \\
- \partial_t^2 a_i^3 \left[ \rho_0 J^{-2,3} + 2 \rho_{0,3} J^{-2} \right] - 2 \partial_t a_i^3 \partial_t \left[ \rho_0 J^{-2,3} + 2 \rho_{0,3} J^{-2} \right].
\end{align*}
\]

The same argument used in the proof of Proposition 9.2 yields the desired results. \( \Box \)

**Proposition 9.5.** For \( t \in [0, T] \), it holds that
\[ \sup_{[0,T]} \left[ \| \eta(t) \|^2_{L^2} + \| \rho_0 J^{-2} (t) \|^2_{L^2} \right] \leq M_0 + \delta \sup_{[0,T]} E(t) + CTP(\sup_{[0,T]} E(t)). \]

**Proof.** We use directly the identity (9.1), i.e.,
\[ \rho_0 a_i^3 J^{-2,3} + 2a_i^3 \rho_{0,3} J^{-2} = -v^i - \rho_0 a_i^\beta J^{-2,\beta} - 2a_i^\beta \rho_{0,\beta} J^{-2}. \]

The same argument used in the proof of Proposition 9.2 implies the desired results. \( \Box \)
10. The A Priori Bound

Combining the inequalities provided by energy estimates, the additional elliptic estimates and the curl estimates shows that

$$\sup_{[0,T]} E(t) \leq M_0 + CTP \left( \sup_{[0,T]} E(t) \right).$$

According to the polynomial-type inequality (2.3), by taking $T > 0$ sufficiently small, we obtain the a priori bound

$$\sup_{[0,T]} E(t) \leq 2M_0.$$

Therefore, we complete the proof of the main theorem.

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REFERENCES


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